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A new kind of signed permutations in the view of context-free grammar

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ABSTRACT

A new kind of signed permutations is proposed in this paper. Based on the standard permutation defined by $\{1, 2, \ldots, n\}$, we assign a sign (either positive or negative) to each number, including the boundary elements at both ends, the resulting permutation is called a signed permutation. Eight combinatorial statistics about ascents, descents, sign changes with respect to the signs are introduced. With the help of the context-free grammar theory, a series of combinatorial properties of such signed permutations are presented, mainly in the aspects of (homogeneous) γ-positivity, (homogeneous) bi-γ-positivity, determinantal expressions, Pólya-frequency, unimodal, log-concave, real-rooted, asymptotic normality.

1. Introduction

This paper introduces and undertakes a detailed study of a new class of signed permutations. It is obtained from a standard permutation by assigning to each entry a sign $(+)$ or $(-)$, including the boundary elements at both ends.

Let $\pi = \pi_1\pi_2 \ldots \pi_n$ be a standard permutation of $[n] = \{1, 2, \ldots, n\}$ with boundary elements $\pi_0 = \pi_{n+1} = 0$. A *signed permutation* of $[n]$ associated with π is a sequence $\pi^{\mathcal{S}} = \pi_1^{\mathcal{S}}\pi_2^{\mathcal{S}} \ldots \pi_n^{\mathcal{S}}$ such that $|\pi_i^{\mathcal{S}}| = \pi_i$ for each $i \in [n]$.

Since the boundary elements π_0 and π_{n+1} are both 0 and carry no sign, we define the signed boundary elements 0^+ and 0^- as formal symbols representing numbers infinitesimally close to 0 from the positive and negative sides, respectively. These symbols are used to extend a signed permutation $\pi^{\mathcal{S}}$ to its boundaries, so that $\pi_0^{\mathcal{S}}, \pi_{n+1}^{\mathcal{S}} \in \{0^+, 0^-\}$. For the sake of consistency in the permutation structure with respect to signs, we formally regard 0^+ and 0^- as having absolute value 0, that is, $|0^+| = |0^-| = 0$.

There are four possible sign patterns for the boundary elements $\pi_0^{\mathcal{S}}$ and $\pi_{n+1}^{\mathcal{S}}$, namely

$$(\pi_0^{\mathcal{S}}, \pi_{n+1}^{\mathcal{S}}) \in \{(0^+, 0^+), (0^+, 0^-), (0^-, 0^+), (0^-, 0^-)\}.$$

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We call a signed permutation $\pi^{\mathcal{S}}$ a *boundary-same-signed signed permutation* if $\pi_0^{\mathcal{S}}\pi_{n+1}^{\mathcal{S}} > 0$, that is, if its boundary signs are either $(0^+, 0^+)$ or $(0^-, 0^-)$. Similarly, we call $\pi^{\mathcal{S}}$ *boundary-opposite-signed* if $\pi_0^{\mathcal{S}}\pi_{n+1}^{\mathcal{S}} < 0$, that is, if its boundary signs are $(0^+, 0^-)$ or $(0^-, 0^+)$.

For consistency with the notation 0^-, for any positive integer i we denote its negative by i^- rather than $-i$. Accordingly, we naturally require the ordering

$$\cdots < 2^- < 1^- < 0^- < 0 < 0^+ < 1 < 2 < \cdots .$$

We also define the corresponding sets as

$$[n]_0 := \{0^+, 1, 2, \ldots, n\}, \qquad [n^-] := \{1^-, 2^-, \ldots, n^-\}, \qquad [n^-]_0 := \{0^-, 1^-, 2^-, \ldots, n^-\}.$$

It is worth noting that our definition of $[n]_0$ differs from the usual convention $[n]_0 = \{0\} \cup [n]$, since here $0^+ \neq 0$.

Let us present two illustrative examples of signed permutations.

- The signed permutation $3\,1^-\,4^-\,2\,5^-$ of [5] is a boundary-same-signed signed permutation if its two boundary elements are both 0^+ or both 0^-.
- The signed permutation $4\,2^-\,6\,5\,1\,3^-$ of [6] is a boundary-opposite-signed signed permutation if one of its boundary elements is 0^+ and the other is 0^-.

Given a signed permutation $\pi^{\mathcal{S}} = \pi_1^{\mathcal{S}}\pi_2^{\mathcal{S}}\ldots\pi_n^{\mathcal{S}}$ of $[n]$ with boundary elements $\pi_0^{\mathcal{S}}$ and $\pi_{n+1}^{\mathcal{S}}$, we define eight combinatorial statistics for it:

$$\begin{aligned}
&\mathrm{pos}(\pi^{\mathcal{S}}) := \#\{i \in \{0\} \cup [n+1] \mid \pi_i^{\mathcal{S}} \in [n]_0\},\\
&\mathrm{neg}(\pi^{\mathcal{S}}) := \#\{i \in \{0\} \cup [n+1] \mid \pi^{\mathcal{S}}(i) \in [n^-]_0\},\\
&\mathrm{asc}^+(\pi^{\mathcal{S}}) := \#\{i \in \{0\} \cup [n] \mid \pi_i^{\mathcal{S}} < \pi_{i+1}^{\mathcal{S}} \text{ and } \pi_i^{\mathcal{S}}, \pi_{i+1}^{\mathcal{S}} \in [n]_0\},\\
&\mathrm{asc}^-(\pi^{\mathcal{S}}) := \#\{i \in \{0\} \cup [n] \mid \pi_i^{\mathcal{S}} < \pi_{i+1}^{\mathcal{S}} \text{ and } \pi_i^{\mathcal{S}}, \pi_{i+1}^{\mathcal{S}} \in [n^-]_0\},\\
&\mathrm{des}^+(\pi^{\mathcal{S}}) := \#\{i \in \{0\} \cup [n] \mid \pi_i^{\mathcal{S}} > \pi_{i+1}^{\mathcal{S}} \text{ and } \pi_i^{\mathcal{S}}, \pi_{i+1}^{\mathcal{S}} \in [n]_0\},\\
&\mathrm{des}^-(\pi^{\mathcal{S}}) := \#\{i \in \{0\} \cup [n] \mid \pi_i^{\mathcal{S}} > \pi_{i+1}^{\mathcal{S}} \text{ and } \pi_i^{\mathcal{S}}, \pi_{i+1}^{\mathcal{S}} \in [n^-]_0\},\\
&\mathrm{sc}^+(\pi^{\mathcal{S}}) := \#\{i \in \{0\} \cup [n] \mid \pi_i^{\mathcal{S}} \in [n]_0, \pi_{i+1}^{\mathcal{S}} \in [n^-]_0\},\\
&\mathrm{sc}^-(\pi^{\mathcal{S}}) := \#\{i \in \{0\} \cup [n] \mid \pi_i^{\mathcal{S}} \in [n^-]_0, \pi_{i+1}^{\mathcal{S}} \in [n]_0\}.
\end{aligned}$$

For convenience, we give the designations for the indices occurring in the above combinatorial statistics:

- each index i counted in $\mathrm{pos}(\pi^{\mathcal{S}})$ ($\mathrm{neg}(\pi^{\mathcal{S}})$, respectively) is called a positive (negative, respectively) index in $\pi^{\mathcal{S}}$;
- each index i counted in $\mathrm{asc}^+(\pi^{\mathcal{S}})$ ($\mathrm{asc}^-(\pi^{\mathcal{S}})$, respectively) is called a positive (negative, respectively) ascent of $\pi^{\mathcal{S}}$;
- each index i counted in $\mathrm{des}^+(\pi^{\mathcal{S}})$ ($\mathrm{des}^-(\pi^{\mathcal{S}})$, respectively) is called a positive (negative, respectively) descent of $\pi^{\mathcal{S}}$;
- each index i counted in $\mathrm{sc}^+(\pi^{\mathcal{S}})$ ($\mathrm{sc}^-(\pi^{\mathcal{S}})$, respectively) is called a positive (negative, respectively) sign change of $\pi^{\mathcal{S}}$.

Let $\mathcal{SP}_n$ denote the set of all signed permutations of $[n]$, where the boundary elements $\pi_0^{\mathcal{S}}$ and $\pi_{n+1}^{\mathcal{S}}$ range over all four possible sign patterns. We define the following F-multivariable polynomials:

$$\begin{aligned}
&F_n(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)\\
&= \sum_{\pi^{\mathcal{S}} \in \mathcal{SP}_n} x_1^{\mathrm{pos}(\pi^{\mathcal{S}})} x_2^{\mathrm{neg}(\pi^{\mathcal{S}})} x_3^{\mathrm{asc}^+(\pi^{\mathcal{S}})} x_4^{\mathrm{asc}^-(\pi^{\mathcal{S}})} x_5^{\mathrm{des}^+(\pi^{\mathcal{S}})} x_6^{\mathrm{des}^-(\pi^{\mathcal{S}})} x_7^{\mathrm{sc}^+(\pi^{\mathcal{S}})} x_8^{\mathrm{sc}^-(\pi^{\mathcal{S}})}.
\end{aligned}$$

Further, we define

$$\mathrm{asc}(\pi^{\mathcal{S}}) = \mathrm{asc}^+(\pi^{\mathcal{S}}) + \mathrm{asc}^-(\pi^{\mathcal{S}}), \qquad \mathrm{des}(\pi^{\mathcal{S}}) = \mathrm{des}^+(\pi^{\mathcal{S}}) + \mathrm{des}^-(\pi^{\mathcal{S}}), \qquad \mathrm{sc}(\pi^{\mathcal{S}}) = \mathrm{sc}^+(\pi^{\mathcal{S}}) + \mathrm{sc}^-(\pi^{\mathcal{S}}),$$

which respectively count the numbers of ascents, descents, and sign changes of $\pi^{\mathcal{S}}$.

To reduce the number of variables and obtain a more compact generating polynomial, we merge the parameters that record positive and negative contributions of the same statistic. In particular, by identifying the pairs (x_3, x_4), (x_5, x_6), and (x_7, x_8), we consider the following specialization of F_n:

$$
\begin{aligned}
&G_n(x_1,x_2,x_3,x_4,x_5)\\
:=\ &F_n(x_1,x_2,x_3,x_3,x_4,x_4,x_5,x_5)\\
=\ &\sum_{\pi^{\mathcal{S}}\in\mathcal{SP}_n} x_1^{\mathrm{pos}(\pi^{\mathcal{S}})}x_2^{\mathrm{neg}(\pi^{\mathcal{S}})}x_3^{\mathrm{asc}^+(\pi^{\mathcal{S}})}x_3^{\mathrm{asc}^-(\pi^{\mathcal{S}})}x_4^{\mathrm{des}^+(\pi^{\mathcal{S}})}x_4^{\mathrm{des}^-(\pi^{\mathcal{S}})}x_5^{\mathrm{sc}^+(\pi^{\mathcal{S}})}x_5^{\mathrm{sc}^-(\pi^{\mathcal{S}})}\\
=\ &\sum_{\pi^{\mathcal{S}}\in\mathcal{SP}_n} x_1^{\mathrm{pos}(\pi^{\mathcal{S}})}x_2^{\mathrm{neg}(\pi^{\mathcal{S}})}x_3^{\mathrm{asc}(\pi^{\mathcal{S}})}x_4^{\mathrm{des}(\pi^{\mathcal{S}})}x_5^{\mathrm{sc}(\pi^{\mathcal{S}})}.
\end{aligned}
$$

It is straightforward to see that

$$\mathrm{asc}(\pi^{\mathcal{S}})+\mathrm{des}(\pi^{\mathcal{S}})+\mathrm{sc}(\pi^{\mathcal{S}})=n+1,$$

for any $\pi^{\mathcal{S}}\in\mathcal{SP}_n$. Therefore, we further define two generating polynomials by appropriate specializations of the variables in G_n:

$$
\begin{aligned}
&H_n(x_1,x_2,x_3,x_4)\\
:=\ &G_n(x_1,x_2,x_3,x_3,x_4)\\
=\ &\sum_{\pi^{\mathcal{S}}\in\mathcal{SP}_n} x_1^{\mathrm{pos}(\pi^{\mathcal{S}})}x_2^{\mathrm{neg}(\pi^{\mathcal{S}})}x_3^{\mathrm{asc}(\pi^{\mathcal{S}})}x_3^{\mathrm{des}(\pi^{\mathcal{S}})}x_4^{\mathrm{sc}(\pi^{\mathcal{S}})}\\
=\ &\sum_{\pi^{\mathcal{S}}\in\mathcal{SP}_n} x_1^{\mathrm{pos}(\pi^{\mathcal{S}})}x_2^{\mathrm{neg}(\pi^{\mathcal{S}})}x_3^{n+1-\mathrm{sc}(\pi^{\mathcal{S}})}x_4^{\mathrm{sc}(\pi^{\mathcal{S}})}
\end{aligned}
$$

and

$$L_n(x,y):=H_n(1,1,y,x)=\sum_{\pi^{\mathcal{S}}\in\mathcal{SP}_n} x^{\mathrm{sc}(\pi^{\mathcal{S}})}y^{n+1-\mathrm{sc}(\pi^{\mathcal{S}})}.$$

For convenience, we write the multivariate polynomials defined above as F_n, G_n, H_n, L_n, if there is no need to indicate the variables in them.

The purpose of this paper is to present some combinatorial properties of the signed permutation $\pi^{\mathcal{S}}$ with respect to the combinatorial statistics $\mathrm{pos}(\pi^{\mathcal{S}})$, $\mathrm{neg}(\pi^{\mathcal{S}})$, $\mathrm{asc}^+(\pi^{\mathcal{S}})$, $\mathrm{asc}^-(\pi^{\mathcal{S}})$, $\mathrm{des}^+(\pi^{\mathcal{S}})$, $\mathrm{des}^-(\pi^{\mathcal{S}})$, $\mathrm{sc}^+(\pi^{\mathcal{S}})$, $\mathrm{sc}^-(\pi^{\mathcal{S}})$. Our paper is organized as follows:

- The context-free grammar technique proposed by Chen [3] plays an important role in our deductions, we will recall it at the beginning of Section 2.
- By establishing the corresponding context-free grammars, in Section 2 we will give the recurrence relations of the coefficients of the multivariate polynomials F_n, G_n, H_n, L_n.
- In a signed permutation, each element carries a positive or negative sign in addition to its position. Consequently, a fundamental feature of signed permutations is the pattern of sign changes. Motivated by this, in the subsequent sections, we focus on the bivariate polynomial $L_n(x,y)$, which encodes the number of sign changes in signed permutations, highlights their combinatorial significance, and provides a convenient tool for further analysis.
- In the subsequent three sections, we investigate the properties of $L_n(x,y)$, presenting several combinatorial aspects related to its coefficients, including (homogeneous) γ-positivity, (homogeneous) bi-γ-positivity, determinantal expressions, Pólya frequency, unimodality, log-concavity, real-rootedness, and asymptotic normality.
- Some concluding remarks and possible research problems are presented in the final section.

2. Context-free grammar theory and its applications to signed permutations

A context-free grammar G over a set $A=\{x,y,z,\dots\}$ consisting of some variables is a set of substitution rules replacing a variable in A by a Laurent polynomial on the variables in A. For a context-free grammar G over A, the formal derivative D (introduced in [3]) with respect to G is defined as a linear operator acting on the Laurent polynomials with variables in A such that each substitution rule is treated as the common differential rule that satisfies the following relations:

$$D(u+v)=D(u)+D(v),\ D(uv)=u\,D(v)+v\,D(u),$$

where $u,v\in A$. In particular, for a constant c, we have $D(c)=0$. Clearly, the Leibniz formula is also valid:

$$D^n(uv)=\sum_{k=0}^{n}\binom{n}{k}D^k(u)D^{n-k}(v).$$

For example, if $G = \{x \to xy, y \to y\}$, then

$$D(x) = xy, D(y) = y, D^2(x) = x(y + y^2), D^3(x) = x(y + 3y^2 + y^3).$$

There are a lot of nice applications of context-free grammars in combinatorics. The readers can refer to [5,10,20,21] for some relevant results. It turns out that many combinatorial structures can be generated by using some particular context-free grammars, such as set partitions [3], permutations [11,22], Stirling permutations [4,25], increasing trees [4,11], rooted trees [12], perfect matchings [25].

It is interesting that for the combinatorial statistics of the signed permutations introduced in Section 1, we can explore them by some particular context-free grammars in the following subsections one by one.

2.1. Context-free grammar and recurrence relation for F_n

Denote by $f_n(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8)$ the coefficient of $x_1^{k_1} x_2^{k_2} x_3^{k_3} x_4^{k_4} x_5^{k_5} x_6^{k_6} x_7^{k_7} x_8^{k_8}$ in the generating polynomial $F_n(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$. Let $A = \{x_3, x_4, x_5, x_6, x_7, x_8\}$, and G be the context-free grammar over A consisting of

$$\begin{aligned}
&x_3 \longrightarrow x_1x_3x_5 + x_2x_7x_8,\ \ x_4 \longrightarrow x_1x_7x_8 + x_2x_4x_6,\\
&x_5 \longrightarrow x_1x_3x_5 + x_2x_7x_8,\ \ x_6 \longrightarrow x_1x_7x_8 + x_2x_4x_6,\\
&x_7 \longrightarrow (x_1x_3 + x_2x_4)x_7,\ \ x_8 \longrightarrow (x_1x_5 + x_2x_6)x_8.
\end{aligned}$$

Let us analyze in detail why this context-free grammar can work for the coefficients $f_n(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8)$.

Given a signed permutation $\pi^{\mathcal{S}} \in \mathcal{SP}_{n-1}$, according to the correspondences between the variables and the combinatorial statistics occurring in F_{n-1}, we can label all the numbers $\pi_i^{\mathcal{S}}$ with $i \in \{0\} \cup [n]$ in $\pi^{\mathcal{S}}$, as well as the positions between $\pi_i^{\mathcal{S}}$ and $\pi_{i+1}^{\mathcal{S}}$ for all $i \in \{0\} \cup [n-1]$:

- If i is a positive (negative, respectively) index of $\pi^{\mathcal{S}}$, then we label x_1 (x_2, respectively) for the number $\pi_i^{\mathcal{S}}$.
- If i is a positive (negative, respectively) ascent of $\pi^{\mathcal{S}}$, then we label x_3 (x_4, respectively) in the position between $\pi_i^{\mathcal{S}}$ and $\pi_{i+1}^{\mathcal{S}}$.
- If i is a positive (negative, respectively) descent of $\pi^{\mathcal{S}}$, then we label x_5 (x_6, respectively) in the position between $\pi_i^{\mathcal{S}}$ and $\pi_{i+1}^{\mathcal{S}}$.
- If i is a positive (negative, respectively) sign change of $\pi^{\mathcal{S}}$, then we label x_7 (x_8, respectively) in the position between $\pi_i^{\mathcal{S}}$ and $\pi_{i+1}^{\mathcal{S}}$.

Further, when we generate another signed permutation $\tilde{\pi}^{\mathcal{S}} \in \mathcal{SP}_n$ based on $\pi^{\mathcal{S}}$, a new number, either n or n^-, is needed to be inserted in $\pi^{\mathcal{S}}$. Assume that the new number is inserted into the position between $\pi_i^{\mathcal{S}}$ and $\pi_{i+1}^{\mathcal{S}}$, where $i \in \{0\} \cup [n-1]$.

If the newly added number is n, then the generating form is given by:

$$\begin{aligned}
&\pi^{\mathcal{S}} = \pi_1^{\mathcal{S}} \cdots \pi_i^{\mathcal{S}}\, \pi_{i+1}^{\mathcal{S}} \cdots \pi_{n-1}^{\mathcal{S}}\\
\to\ &\tilde{\pi}^{\mathcal{S}} = \pi_1^{\mathcal{S}} \cdots \pi_i^{\mathcal{S}}\, n\, \pi_{i+1}^{\mathcal{S}} \cdots \pi_{n-1}^{\mathcal{S}}\\
&\quad = \tilde{\pi}_1^{\mathcal{S}} \cdots \tilde{\pi}_i^{\mathcal{S}}\, \tilde{\pi}_{i+1}^{\mathcal{S}}\, \tilde{\pi}_{i+2}^{\mathcal{S}} \cdots \tilde{\pi}_n^{\mathcal{S}},
\end{aligned}$$

where $\tilde{\pi}_j^{\mathcal{S}} = \pi_j^{\mathcal{S}}$ for any $j \in \{0\} \cup [i]$, $\tilde{\pi}_{i+1}^{\mathcal{S}} = n$, and $\tilde{\pi}_j^{\mathcal{S}} = \pi_{j-1}^{\mathcal{S}}$ for any $j \in \{i+2, i+3, \ldots, n+1\}$.

If the newly added number is n^-, then the case should be

$$\begin{aligned}
&\pi^{\mathcal{S}} = \pi_1^{\mathcal{S}} \cdots \pi_i^{\mathcal{S}}\, \pi_{i+1}^{\mathcal{S}} \cdots \pi_{n-1}^{\mathcal{S}}\\
\to\ &\tilde{\pi}^{\mathcal{S}} = \pi_1^{\mathcal{S}} \cdots \pi_i^{\mathcal{S}}\, n^-\, \pi_{i+1}^{\mathcal{S}} \cdots \pi_{n-1}^{\mathcal{S}}\\
&\quad = \tilde{\pi}_1^{\mathcal{S}} \cdots \tilde{\pi}_i^{\mathcal{S}}\, \tilde{\pi}_{i+1}^{\mathcal{S}}\, \tilde{\pi}_{i+2}^{\mathcal{S}} \cdots \tilde{\pi}_n^{\mathcal{S}},
\end{aligned}$$

where $\tilde{\pi}_j^{\mathcal{S}} = \pi_j^{\mathcal{S}}$ for any $j \in \{0\} \cup [i]$, $\tilde{\pi}_{i+1}^{\mathcal{S}} = n^-$, and $\tilde{\pi}_j^{\mathcal{S}} = \pi_{j-1}^{\mathcal{S}}$ for any $j \in \{i+2, i+3, \ldots, n+1\}$.

First we consider the case that the new number is n (a positive number). From the corresponding generating form, one can see that

- If i is a positive ascent of $\pi^{\mathcal{S}}$, then there is one more positive number and one more positive descent in $\tilde{\pi}^{\mathcal{S}}$ than in $\pi^{\mathcal{S}}$, and the cardinalities of the other statistics remain unchanged, leading to the context-free grammar $x_3 \longrightarrow x_1x_3x_5$. This context-free grammar is also valid when i is a positive descent of $\pi^{\mathcal{S}}$, i.e., $x_5 \longrightarrow x_1x_3x_5$.

- If i is a negative ascent of $\pi^{\mathcal{S}}$, then there is one more positive number, one less negative ascent, one more positive sign change, and one more negative sign change in $\tilde{\pi}^{\mathcal{S}}$ than in $\pi^{\mathcal{S}}$, and the cardinalities of the other statistics remain unchanged, leading to the context-free grammar $x_4 \longrightarrow x_1x_7x_8$. This context-free grammar is also valid when i is a negative descent of $\pi^{\mathcal{S}}$, i.e., $x_6 \longrightarrow x_1x_7x_8$.
- If i is a positive sign change of $\pi^{\mathcal{S}}$, then there is one more positive number and one more positive ascent in $\tilde{\pi}^{\mathcal{S}}$ than in $\pi^{\mathcal{S}}$, and the cardinalities of the other statistics remain unchanged, leading to the context-free grammar $x_7 \longrightarrow x_1x_3x_7$.
- If i is a negative sign change of $\pi^{\mathcal{S}}$, then there is one more positive number and one more positive descent in $\tilde{\pi}^{\mathcal{S}}$ than in $\pi^{\mathcal{S}}$, and the cardinalities of the other statistics remain unchanged, leading to the context-free grammar $x_8 \longrightarrow x_1x_5x_8$.

Next suppose that the new number is n^- (a negative number). Accordingly, we have the following possibilities:

- If i is a positive ascent of $\pi^{\mathcal{S}}$, then there is one more negative number, one less positive ascent, one more positive sign change, and one more negative sign change in $\tilde{\pi}^{\mathcal{S}}$ than in $\pi^{\mathcal{S}}$, and the cardinalities of the other statistics remain unchanged, leading to the context-free grammar $x_3 \longrightarrow x_2x_7x_8$. This context-free grammar is also valid when i is a positive descent of $\pi^{\mathcal{S}}$, i.e., $x_5 \longrightarrow x_2x_7x_8$.
- If i is a negative ascent of $\pi^{\mathcal{S}}$, then there is one more negative number and one more negative descent in $\tilde{\pi}^{\mathcal{S}}$ than in $\pi^{\mathcal{S}}$, and the cardinalities of the other statistics remain unchanged, leading to the context-free grammar $x_4 \longrightarrow x_2x_4x_6$. This context-free grammar is also valid when i is a negative descent of $\pi^{\mathcal{S}}$, i.e., $x_6 \longrightarrow x_2x_4x_6$.
- If i is a positive sign change of $\pi^{\mathcal{S}}$, then there is one more negative number and one more negative ascent in $\tilde{\pi}^{\mathcal{S}}$ than in $\pi^{\mathcal{S}}$, and the cardinalities of the other statistics remain unchanged, leading to the context-free grammar $x_7 \longrightarrow x_2x_4x_7$.
- If i is a negative sign change of $\pi^{\mathcal{S}}$, then there is one more negative number and one more negative descent in $\tilde{\pi}^{\mathcal{S}}$ than in $\pi^{\mathcal{S}}$, and the cardinalities of the other statistics remain unchanged, leading to the context-free grammar $x_8 \longrightarrow x_2x_6x_8$.

Combining the above two cases (adding n or n^-), we can make the conclusion:

- If i is a positive ascent of $\pi^{\mathcal{S}}$, then $x_3 \longrightarrow x_1x_3x_5 + x_2x_7x_8$, where the first term on the right-hand side corresponds to the case when the new number is n, while the second term corresponds to the case when the new number is n^-, the same holds for the other situations listed below.
- If i is a negative ascent of $\pi^{\mathcal{S}}$, then $x_4 \longrightarrow x_1x_7x_8 + x_2x_4x_6$.
- If i is a positive descent of $\pi^{\mathcal{S}}$, then $x_5 \longrightarrow x_1x_3x_5 + x_2x_7x_8$.
- If i is a negative descent of $\pi^{\mathcal{S}}$, then $x_6 \longrightarrow x_1x_7x_8 + x_2x_4x_6$.
- If i is a positive sign change of $\pi^{\mathcal{S}}$, then $x_7 \longrightarrow (x_1x_3 + x_2x_4)x_7$.
- If i is a negative sign change of $\pi^{\mathcal{S}}$, then $x_8 \longrightarrow (x_1x_5 + x_2x_6)x_8$.

Theorem 2.1. *The coefficients of F_n satisfy the recurrence relation*

$$\begin{aligned}
f_n(k_1,\ldots,k_8) = {} & k_3\, f_{n-1}\big(k_1-1,k_2,k_3,k_4,k_5-1,k_6,k_7,k_8\big)\\
&+ (k_3+1)\, f_{n-1}\big(k_1,k_2-1,k_3+1,k_4,k_5,k_6,k_7-1,k_8-1\big)\\
&+ (k_4+1)\, f_{n-1}\big(k_1-1,k_2,k_3,k_4+1,k_5,k_6,k_7-1,k_8-1\big)\\
&+ k_4\, f_{n-1}\big(k_1,k_2-1,k_3,k_4,k_5,k_6-1,k_7,k_8\big)\\
&+ k_5\, f_{n-1}\big(k_1-1,k_2,k_3-1,k_4,k_5,k_6,k_7,k_8\big)\\
&+ (k_5+1)\, f_{n-1}\big(k_1,k_2-1,k_3,k_4,k_5+1,k_6,k_7-1,k_8-1\big)\\
&+ (k_6+1)\, f_{n-1}\big(k_1-1,k_2,k_3,k_4,k_5,k_6+1,k_7-1,k_8-1\big)\\
&+ k_6\, f_{n-1}\big(k_1,k_2-1,k_3,k_4-1,k_5,k_6,k_7,k_8\big)\\
&+ k_7\, f_{n-1}\big(k_1-1,k_2,k_3-1,k_4,k_5,k_6,k_7,k_8\big)\\
&+ k_7\, f_{n-1}\big(k_1,k_2-1,k_3,k_4-1,k_5,k_6,k_7,k_8\big)\\
&+ k_8\, f_{n-1}\big(k_1-1,k_2,k_3,k_4,k_5-1,k_6,k_7,k_8\big)\\
&+ k_8\, f_{n-1}\big(k_1,k_2-1,k_3,k_4,k_5,k_6-1,k_7,k_8\big).
\end{aligned}$$

Proof. From the context-free grammar

$$x_3 \longrightarrow x_1x_3x_5 + x_2x_7x_8, \quad x_4 \longrightarrow x_1x_7x_8 + x_2x_4x_6, \quad x_5 \longrightarrow x_1x_3x_5 + x_2x_7x_8,$$

$$x_6 \longrightarrow x_1x_7x_8 + x_2x_4x_6, \quad x_7 \longrightarrow (x_1x_3 + x_2x_4)x_7, \quad x_8 \longrightarrow (x_1x_5 + x_2x_6)x_8,$$

we have

$$\begin{aligned}
&F_n(x_1, \ldots, x_8) \\
&= \sum_{k_1,\ldots,k_8=0}^{n+1} f_n(k_1, \ldots, k_8)\, x_1^{k_1} x_2^{k_2} x_3^{k_3} x_4^{k_4} x_5^{k_5} x_6^{k_6} x_7^{k_7} x_8^{k_8} \\
&= D(F_{n-1}(x_1, \ldots, x_8)) \\
&= D\left(\sum_{t_1,\ldots,t_8=0}^{n} f_{n-1}(t_1, \ldots, t_8)\, x_1^{t_1} x_2^{t_2} x_3^{t_3} x_4^{t_4} x_5^{t_5} x_6^{t_6} x_7^{t_7} x_8^{t_8} \right) \\
&= \sum_{t_1,\ldots,t_8=0}^{n} f_{n-1}(t_1, \ldots, t_8) \Big[t_3\, x_1^{t_1+1} x_2^{t_2} x_3^{t_3} x_4^{t_4} x_5^{t_5+1} x_6^{t_6} x_7^{t_7} x_8^{t_8} \\
&\qquad + t_3\, x_1^{t_1} x_2^{t_2+1} x_3^{t_3-1} x_4^{t_4} x_5^{t_5} x_6^{t_6} x_7^{t_7+1} x_8^{t_8+1} \\
&\qquad + t_4\, x_1^{t_1+1} x_2^{t_2} x_3^{t_3} x_4^{t_4-1} x_5^{t_5} x_6^{t_6} x_7^{t_7+1} x_8^{t_8+1} \\
&\qquad + t_4\, x_1^{t_1} x_2^{t_2+1} x_3^{t_3} x_4^{t_4} x_5^{t_5} x_6^{t_6+1} x_7^{t_7} x_8^{t_8} \\
&\qquad + t_5\, x_1^{t_1+1} x_2^{t_2} x_3^{t_3+1} x_4^{t_4} x_5^{t_5} x_6^{t_6} x_7^{t_7} x_8^{t_8} \\
&\qquad + t_5\, x_1^{t_1} x_2^{t_2+1} x_3^{t_3} x_4^{t_4} x_5^{t_5-1} x_6^{t_6} x_7^{t_7+1} x_8^{t_8+1} \\
&\qquad + t_6\, x_1^{t_1+1} x_2^{t_2} x_3^{t_3} x_4^{t_4} x_5^{t_5} x_6^{t_6-1} x_7^{t_7+1} x_8^{t_8+1} \\
&\qquad + t_6\, x_1^{t_1} x_2^{t_2+1} x_3^{t_3} x_4^{t_4+1} x_5^{t_5} x_6^{t_6} x_7^{t_7} x_8^{t_8} \\
&\qquad + t_7\, x_1^{t_1+1} x_2^{t_2} x_3^{t_3+1} x_4^{t_4} x_5^{t_5} x_6^{t_6} x_7^{t_7} x_8^{t_8} \\
&\qquad + t_7\, x_1^{t_1} x_2^{t_2+1} x_3^{t_3} x_4^{t_4+1} x_5^{t_5} x_6^{t_6} x_7^{t_7} x_8^{t_8} \\
&\qquad + t_8\, x_1^{t_1+1} x_2^{t_2} x_3^{t_3} x_4^{t_4} x_5^{t_5+1} x_6^{t_6} x_7^{t_7} x_8^{t_8} \\
&\qquad + t_8\, x_1^{t_1} x_2^{t_2+1} x_3^{t_3} x_4^{t_4} x_5^{t_5} x_6^{t_6+1} x_7^{t_7} x_8^{t_8} \Big].
\end{aligned}$$

After extracting the coefficients of $x_1^{k_1} x_2^{k_2} x_3^{k_3} x_4^{k_4} x_5^{k_5} x_6^{k_6} x_7^{k_7} x_8^{k_8}$ for both sides, the desired recurrence relation follows. □

For other multivariate polynomials G_n, H_n, L_n, each one is just a special case of F_n (with some specific variables), thus the corresponding grammars and recurrence relations can be obtained in a similar way, we omit the detailed proofs in subsequent subsections to avoid the unnecessary repetition.

2.2. Context-free grammar and recurrence relation for G_n

Denote by $g_n(k_1, k_2, k_3, k_4, k_5)$ the coefficient of $x_1^{k_1} x_2^{k_2} x_3^{k_3} x_4^{k_4} x_5^{k_5}$ in $G_n(x_1, x_2, x_3, x_4, x_5)$. Let $A = \{x_3, x_4, x_5\}$, and G be the contex-free grammar over A consisting of

$$x_3 \longrightarrow (x_1 + x_2)(x_3x_4 + x_5^2), \quad x_4 \longrightarrow (x_1 + x_2)(x_3x_4 + x_5^2), \quad x_5 \longrightarrow (x_1 + x_2)(x_3 + x_4)x_5.$$

Theorem 2.2. *The coefficients of G_n satisfy the recurrence relation*

$$\begin{aligned}
g_n(k_1, k_2, k_3, k_4, k_5) = {} & k_3\, g_{n-1}\big(k_1-1, k_2, k_3, k_4-1, k_5\big) + k_3\, g_{n-1}\big(k_1, k_2-1, k_3, k_4-1, k_5\big) \\
& + (k_3+1)\Big[g_{n-1}\big(k_1-1, k_2, k_3+1, k_4, k_5-2\big) + g_{n-1}\big(k_1, k_2-1, k_3+1, k_4, k_5-2\big) \Big] \\
& + k_4 \Big[g_{n-1}\big(k_1-1, k_2, k_3-1, k_4, k_5\big) + g_{n-1}\big(k_1, k_2-1, k_3-1, k_4, k_5\big) \Big] \\
& + (k_4+1)\Big[g_{n-1}\big(k_1-1, k_2, k_3, k_4+1, k_5-2\big) + g_{n-1}\big(k_1, k_2-1, k_3, k_4+1, k_5-2\big) \Big] \\
& + k_5 \Big[g_{n-1}\big(k_1-1, k_2, k_3-1, k_4, k_5\big) + g_{n-1}\big(k_1-1, k_2, k_3, k_4-1, k_5\big) \\
& + g_{n-1}\big(k_1, k_2-1, k_3-1, k_4, k_5\big) + g_{n-1}\big(k_1, k_2-1, k_3, k_4-1, k_5\big) \Big].
\end{aligned}$$

2.3. Context-free grammar and recurrence relation for H_n

Denote by $h_n(k_1,k_2,k_3,k_4)$ the coefficient of $x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}$ in $H_n(x_1,x_2,x_3,x_4)$. Let $A=\{x_3,x_4\}$, and G be the contex-free grammar over A consisting of

$$x_3 \longrightarrow (x_1+x_2)(x_3^2+x_4^2),\ x_4 \longrightarrow (x_1+x_2)x_3x_4.$$

Theorem 2.3. *The coefficients of H_n satisfy the recurrence relation*

$$\begin{aligned} h_n(k_1,k_2,k_3,k_4) = {} & (k_3-1)\,h_{n-1}\big(k_1-1,k_2,k_3-1,k_4\big) + (k_3-1)\,h_{n-1}\big(k_1,k_2-1,k_3-1,k_4\big) \\ & + (k_3+1)\,h_{n-1}\big(k_1-1,k_2,k_3+1,k_4-2\big) + (k_3+1)\,h_{n-1}\big(k_1,k_2-1,k_3+1,k_4-2\big) \\ & + k_4\,h_{n-1}\big(k_1-1,k_2,k_3-1,k_4\big) + k_4\,h_{n-1}\big(k_1,k_2-1,k_3-1,k_4\big). \end{aligned}$$

2.4. Context-free grammar and recurrence relation for L_n

Denote by $\ell_n(k_1,k_2)$ the coefficient of $x^{k_1}y^{k_2}$ in $L_n(x,y)$. Let $A=\{x,y\}$, and G be the contex-free grammar over A consisting of

$$x \longrightarrow 2xy,\quad y \longrightarrow x^2+y^2.$$

Theorem 2.4. *The coefficients of L_n satisfy the recurrence relation*

$$\ell_n(k_1,k_2) = (2k_1+k_2-1)\ell_{n-1}(k_1,k_2-1) + (k_2+1)\ell_{n-1}(k_1-2,k_2+1).$$

From the expression of $L_n(x,y)$, that is

$$L_n(x,y) = \sum_{\pi^{\mathcal{S}}\in\mathcal{SP}_n} x^{\mathrm{sc}(\pi^{\mathcal{S}})}y^{n+1-\mathrm{sc}(\pi^{\mathcal{S}})},$$

we can rewrite it as

$$L_n(x,y) = \sum_{k=0}^{n+1} c(n,k)x^k y^{n+1-k},$$

where $c(n,k)$ denotes the number of signed permutations of $[n]$ with exactly k sign changes, with the boundary elements $\pi_0^{\mathcal{S}}$ and $\pi_{n+1}^{\mathcal{S}}$ ranging over all four possible sign patterns. Accordingly, we can obtain the recurrence relation of $c(n,k)$ by rephrasing Theorem 2.4. It is also of interest to give a combinatorial proof for the recurrence relation of $c(n,k)$.

Theorem 2.5. *The numbers $c(n,k)$ satisfy the recurrence relation*

$$c(n,k) = (n+k)\,c(n-1,k) + (n+2-k)\,c(n-1,k-2), \qquad n\geqslant 2,$$

together with the initial conditions

$$c(1,0)=2,\quad c(1,1)=4,\quad c(1,2)=2,\quad c(1,k)=0 \textit{ for all other } k.$$

Proof. The initial conditions for $c(n,k)$ follow easily from

$$\mathcal{SP}_1 = \{0^+10^+,\ 0^+10^-,\ 0^-10^+,\ 0^-10^-,\ 0^+1^-0^+,\ 0^+1^-0^-,\ 0^-1^-0^+,\ 0^-1^-0^-\}.$$

Next, we derive the recurrence relation for $c(n,k)$.

Assume that $\pi^{\mathcal{S}}=\pi_1^{\mathcal{S}}\cdots\pi_n^{\mathcal{S}}$ is a signed permutation of $[n]$ with $\mathrm{sc}(\pi^{\mathcal{S}})=k$. Either n or n^- is in $\pi^{\mathcal{S}}$, assume that $\pi_{i+1}^{\mathcal{S}}=n$ or $\pi_{i+1}^{\mathcal{S}}=n^-$, where $i\in\{0\}\cup[n-1]$. After deleting n or n^- in $\pi^{\mathcal{S}}$, the resulting signed permutation of $[n-1]$ is denoted by $\tilde{\pi}^{\mathcal{S}}=\tilde{\pi}_1^{\mathcal{S}}\cdots\tilde{\pi}_{n-1}^{\mathcal{S}}$. We claim that either $\mathrm{sc}(\tilde{\pi}^{\mathcal{S}})=k$ or $\mathrm{sc}(\tilde{\pi}^{\mathcal{S}})=k-2$.

First suppose that $\pi_{i+1}^{\mathcal{S}}=n$. There are exactly three possibilities to calculate the differences between $\mathrm{sc}(\tilde{\pi}^{\mathcal{S}})$ and $\mathrm{sc}(\pi^{\mathcal{S}})$:

- When $\pi_i^{\mathcal{S}}$ and $\pi_{i+2}^{\mathcal{S}}$ are both positive (equivalently, $\tilde{\pi}_i^{\mathcal{S}}$ and $\tilde{\pi}_{i+1}^{\mathcal{S}}$ are both positive, thus i is either a positive ascent or a positive descent), we have $\mathrm{sc}(\tilde{\pi}^{\mathcal{S}})=\mathrm{sc}(\pi^{\mathcal{S}})$.
- When $\pi_i^{\mathcal{S}}$ and $\pi_{i+2}^{\mathcal{S}}$ are both negative (equivalently, $\tilde{\pi}_i^{\mathcal{S}}$ and $\tilde{\pi}_{i+1}^{\mathcal{S}}$ are both negative, thus i is either a negative ascent or a negative descent), $\mathrm{sc}(\tilde{\pi}^{\mathcal{S}})=\mathrm{sc}(\pi^{\mathcal{S}})-2$.

- When $\pi_i^{\mathcal{S}}$ and $\pi_{i+2}^{\mathcal{S}}$ have different signs (equivalently, $\tilde{\pi}_i^{\mathcal{S}}$ and $\tilde{\pi}_{i+1}^{\mathcal{S}}$ have different signs, thus i is either a positive sign change or a negative sign change), $\mathrm{sc}(\tilde{\pi}^{\mathcal{S}}) = \mathrm{sc}(\pi^{\mathcal{S}})$.

Next suppose that $\pi_{i+1}^{\mathcal{S}} = n^-$. Similarly, we have

- When $\pi_i^{\mathcal{S}}$ and $\pi_{i+2}^{\mathcal{S}}$ are both positive (equivalently, $\tilde{\pi}_i^{\mathcal{S}}$ and $\tilde{\pi}_{i+1}^{\mathcal{S}}$ are both positive, thus i is either a positive ascent or a positive descent), $\mathrm{sc}(\tilde{\pi}^{\mathcal{S}}) = \mathrm{sc}(\pi^{\mathcal{S}}) - 2$.
- When $\pi_i^{\mathcal{S}}$ and $\pi_{i+2}^{\mathcal{S}}$ are both negative (equivalently, $\tilde{\pi}_i^{\mathcal{S}}$ and $\tilde{\pi}_{i+1}^{\mathcal{S}}$ are both negative, thus i is either a negative ascent or a negative descent), $\mathrm{sc}(\tilde{\pi}^{\mathcal{S}}) = \mathrm{sc}(\pi^{\mathcal{S}})$.
- When $\pi_i^{\mathcal{S}}$ and $\pi_{i+2}^{\mathcal{S}}$ have different signs (equivalently, $\tilde{\pi}_i^{\mathcal{S}}$ and $\tilde{\pi}_{i+1}^{\mathcal{S}}$ have different signs, thus i is either a positive sign change or a negative sign change), $\mathrm{sc}(\tilde{\pi}^{\mathcal{S}}) = \mathrm{sc}(\pi^{\mathcal{S}})$.

Combining all the possibilities mentioned above, we conclude with the claim that either $\mathrm{sc}(\tilde{\pi}^{\mathcal{S}}) = k$ or $\mathrm{sc}(\tilde{\pi}^{\mathcal{S}}) = k-2$, based on $\mathrm{sc}(\pi^{\mathcal{S}}) = k$.

Furthermore, if $\mathrm{sc}(\tilde{\pi}^{\mathcal{S}}) = k$, then there are $c(n-1,k)$ ways to obtain $\tilde{\pi}^{\mathcal{S}}$, and according to the analysis in last paragraph, the number of positions we can place n or n^- to recover $\pi^{\mathcal{S}}$ with $\mathrm{sc}(\pi^{\mathcal{S}}) = k$ is equal to

$$\begin{aligned}
&\mathrm{asc}^+(\tilde{\pi}^{\mathcal{S}}) + \mathrm{des}^+(\tilde{\pi}^{\mathcal{S}}) + \mathrm{asc}^-(\tilde{\pi}^{\mathcal{S}}) + \mathrm{des}^-(\tilde{\pi}^{\mathcal{S}}) + 2(\mathrm{sc}^+(\tilde{\pi}^{\mathcal{S}}) + \mathrm{sc}^-(\tilde{\pi}^{\mathcal{S}}))\\
&= n + \mathrm{sc}^+(\tilde{\pi}^{\mathcal{S}}) + \mathrm{sc}^-(\tilde{\pi}^{\mathcal{S}})\\
&= n + \mathrm{sc}(\tilde{\pi}^{\mathcal{S}})\\
&= n + k.
\end{aligned}$$

In an analogous fashion, we can deduce that when $\mathrm{sc}(\tilde{\pi}^{\mathcal{S}}) = k-2$, there are $c(n-1,k-2)$ ways to obtain $\tilde{\pi}^{\mathcal{S}}$, and there are

$$\mathrm{asc}^-(\tilde{\pi}^{\mathcal{S}}) + \mathrm{des}^-(\tilde{\pi}^{\mathcal{S}}) + \mathrm{asc}^+(\tilde{\pi}^{\mathcal{S}}) + \mathrm{des}^+(\tilde{\pi}^{\mathcal{S}}) = n - \mathrm{sc}(\tilde{\pi}^{\mathcal{S}}) = n + 2 - k$$

positions of $\tilde{\pi}^{\mathcal{S}}$ we can insert n or n^- to recover $\pi^{\mathcal{S}}$ with $\mathrm{sc}(\pi^{\mathcal{S}}) = k$.

In conclusion, there are

$$(n+k)\,c(n-1,k) + (n+2-k)\,c(n-1,k-2)$$

ways to construct a signed permutation of $[n]$ with exactly k sign changes. $\square$

Solving the recurrence for $c(n,k)$, we obtain the closed-form expression

$$c(n,k) = 2n!\binom{n+1}{k}.$$

Furthermore, we observe that the parity of k is determined by whether the two boundary elements of a signed permutation have the same sign or opposite signs. More precisely, a signed permutation of $[n]$ with an even number of sign changes must have same-signed boundary elements, whereas a signed permutation of $[n]$ with an odd number of sign changes must have opposite-signed boundary elements. Therefore, we can refine the enumeration of $c(n,k)$ in terms of boundary-same-signed and boundary-opposite-signed signed permutations.

Theorem 2.6. *The quantity $c(n,k) = 2n!\binom{n+1}{k}$ enumerates boundary-same-signed (boundary-opposite-signed, respectively) signed permutations of $[n]$ with k sign changes when k is even (odd, respectively).*

Since different parities of k correspond to different combinatorial models, we partition the binomial coefficients $n!\binom{n+1}{k}$, with $k = 0, 1, \ldots, n+1$, into two cases according to the parity of k. Specifically, we define

$$a(n,k) := n!\binom{n+1}{2k+1}, \quad \text{for } n \geqslant 0 \text{ and } k = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor,$$

and

$$b(n,k) := n!\binom{n+1}{2k}, \quad \text{for } n \geqslant 0 \text{ and } k = 0, 1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Accordingly, we partition $\mathcal{SP}_n$ into four classes according to the signs of the boundary elements $\pi_0^{\mathcal{S}}$ and $\pi_{n+1}^{\mathcal{S}}$:

$$\mathcal{SP}_n^{(+,+)}, \quad \mathcal{SP}_n^{(+,-)}, \quad \mathcal{SP}_n^{(-,+)}, \quad \mathcal{SP}_n^{(-,-)}.$$

For instance, $\mathcal{SP}_n^{(+,+)}$ denotes the set of all signed permutations of $[n]$ whose two boundary elements are both 0^+, similarly, the other three classes correspond to the remaining three possible boundary sign patterns. Then for any nonnegative integer n,

$$\begin{aligned}P_n(x,y) &:= \sum_{\pi^{\mathcal{S}}\in\mathcal{SP}_n^{(+,-)}} x^{\mathrm{sc}(\pi^{\mathcal{S}})}y^{n+1-\mathrm{sc}(\pi^{\mathcal{S}})} = \sum_{\pi^{\mathcal{S}}\in\mathcal{SP}_n^{(-,+)}} x^{\mathrm{sc}(\pi^{\mathcal{S}})}y^{n+1-\mathrm{sc}(\pi^{\mathcal{S}})}\\ &= n!\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}x^{2k+1}y^{n-2k} = \sum_{k=0}^{\lfloor n/2\rfloor} a(n,k)\,x^{2k+1}y^{n-2k}\end{aligned}$$

and

$$\begin{aligned}Q_n(x,y) &:= \sum_{\pi^{\mathcal{S}}\in\mathcal{SP}_n^{(+,+)}} x^{\mathrm{sc}(\pi^{\mathcal{S}})}y^{n+1-\mathrm{sc}(\pi^{\mathcal{S}})} = \sum_{\pi^{\mathcal{S}}\in\mathcal{SP}_n^{(-,-)}} x^{\mathrm{sc}(\pi^{\mathcal{S}})}y^{n+1-\mathrm{sc}(\pi^{\mathcal{S}})}\\ &= n!\sum_{k=0}^{\lfloor (n+1)/2\rfloor}\binom{n+1}{2k}x^{2k}y^{n+1-2k} = \sum_{k=0}^{\lfloor (n+1)/2\rfloor} b(n,k)\,x^{2k}y^{n+1-2k}.\end{aligned}$$

For a better illustration, we list $P_n(x,y)$ and $Q_n(x,y)$ with $0 \leqslant n \leqslant 6$ as examples,

n	$P_n(x,y)$	$Q_n(x,y)$
0	x	y
1	$2xy$	x^2+y^2
2	$2x^3+6xy^2$	$6x^2y+2y^3$
3	$24x^3y+24xy^3$	$6x^4+36x^2y^2+6y^4$
4	$24x^5+240x^3y^2+120xy^4$	$120x^4y+240x^2y^3+24y^5$
5	$720x^5y+2400x^3y^3+720xy^5$	$120x^6+1800x^4y^2+1800x^2y^4+120y^6$
6	$720x^7+15120x^5y^2+25200x^3y^4+5040xy^6$	$5040x^6y+25200x^4y^3+15120x^2y^5+720y^7$

Besides the bivariate polynomials $P_n(x,y)$ and $Q_n(x,y)$, we also introduce the univariate generating functions of $a(n,k)$ and $b(n,k)$ in the variable x, defined by

$$P_n(x) := \sum_{k=0}^{\lfloor n/2\rfloor} a(n,k)\,x^k = n!\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}x^k,$$

and

$$Q_n(x) := \sum_{k=0}^{\lfloor (n+1)/2\rfloor} b(n,k)\,x^k = n!\sum_{k=0}^{\lfloor (n+1)/2\rfloor}\binom{n+1}{2k}x^k,$$

for any nonnegative integer n. A direct observation then leads to

$$P_n(x,1) = x\,P_n(x^2), \tag{1}$$

and

$$Q_n(x,1) = Q_n(x^2). \tag{2}$$

We list $P_n(x)$ and $Q_n(x)$ with $0 \leqslant n \leqslant 6$ as examples,

n	$P_n(x)$	$Q_n(x)$
0	1	1
1	2	$1+x$
2	$6+2x$	$2+6x$
3	$24+24x$	$6+36x+6x^2$
4	$120+240x+24x^2$	$24+240x+120x^2$
5	$720+2400x+720x^2$	$120+1800x+1800x^2+120x^3$
6	$5040+25200x+15120x^2+720x^3$	$720+15120x+25200x^2+5040x^3$

It is straightforward to verify that the numbers $a(n,k)$ and $b(n,k)$ satisfy the two-term recurrence relations

$$a(n,k) = (n+2k+1)\,a(n-1,k) + (n-2k+1)\,a(n-1,k-1), \tag{3}$$

$$b(n,k) = (n+2k)\,b(n-1,k) + (n-2k+2)\,b(n-1,k-1), \tag{4}$$

with initial conditions

$$a(0,k) = b(0,k) = 1 \text{ for } k = 0, \quad a(0,k) = b(0,k) = 0 \text{ for all other } k,$$

which provides an equivalent formulation of Theorem 2.5.

We can also give the three-term recurrence relations of $a(n,k)$ and $b(n,k)$ as

$$a(n,k) = 2n \cdot a(n-1,k) - n(n-1)(a(n-2,k) - a(n-2,k-1)) \tag{5}$$

and

$$b(n,k) = 2n \cdot b(n-1,k) - n(n-1)(b(n-2,k) - b(n-2,k-1)). \tag{6}$$

A recurrence relation involving in both of $a(n,k)$ and $b(n,k)$ can be seen as

$$a(n,k) = b(n,k) + n(a(n-1,k) - a(n-1,k-1)). \tag{7}$$

From the recurrence relations of $a(n,k)$ and $b(n,k)$ as presented in (3) and (4), it is not hard to obtain the recurrence relations of $P_n(x)$ and $Q_n(x)$, respectively.

Theorem 2.7. *The univariate polynomials $P_n(x)$ and $Q_n(x)$ satisfy the recurrence relations*

$$P_{n+1}(x) = (nx + n + 2)\, P_n(x) - 2x(x-1)\, P_n'(x),$$

with initial condition $P_0(x) = 1$, and

$$Q_{n+1}(x) = (n+1)(x+1)\, Q_n(x) - 2x(x-1)\, Q_n'(x),$$

with initial condition $Q_0(x) = 1$.

We get the recurrence relations of $P_n(x, y)$ and $Q_n(x, y)$ by virtue of (5) and (6).

Theorem 2.8. *The bivariate polynomials $P_n(x, y)$ and $Q_n(x, y)$ satisfy the recurrence relations*

$$P_n(x, y) = 2ny\, P_{n-1}(x, y) + n(n-1)(x^2 - y^2)\, P_{n-2}(x, y),$$

with initial conditions $P_0(x, y) = x$ and $P_1(x, y) = 2xy$, and

$$Q_n(x, y) = 2ny\, Q_{n-1}(x, y) + n(n-1)(x^2 - y^2)\, Q_{n-2}(x, y),$$

with initial conditions $Q_0(x, y) = y$ and $Q_1(x, y) = x^2 + y^2$.

The derivation of another recurrence relation involving both $P_n(x, y)$ and $Q_n(x, y)$ makes use of (7).

Theorem 2.9. *The bivariate polynomials $P_n(x, y)$ and $Q_n(x, y)$ satisfy the recurrence relation*

$$y \cdot P_n(x, y) = x \cdot Q_n(x, y) - n(x^2 - y^2)P_{n-1}(x, y).$$

2.5. Context-free grammar for $a(n,k)$ and $b(n,k)$

As what we stated above about $L_n(x, y)$, it is known that $a(n,k)$ and $b(n,k)$ can be generated by the same grammar

$$G = \{x \to 2xy,\, y \to x^2 + y^2\}.$$

Theorem 2.10. *Let $G = \{x \to 2xy, y \to x^2 + y^2\}$. Then*

$$D^n(x) = P_n(x, y), \quad D^n(y) = Q_n(x, y).$$

It is worth indicating that $G = \{x \to 2xy, y \to x^2 + y^2\}$ is not the unique grammar which is capable of generating $a(n,k)$ and $b(n,k)$. In the sequel, we provide an alternative grammar that simultaneously generates $a(n,k)$ and $b(n,k)$.

Theorem 2.11. *Let $G = \{u \to v^3, v \to uv^2\}$. Then*

$$D^n(v^2) = \sum_{k=0}^{\lfloor n/2 \rfloor} a(n,k)\, u^{n-2k} v^{n+2+2k} = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} u^{n-2k} v^{n+2+2k}$$

and

$$D^n(uv) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} b(n,k)\, u^{n+1-2k} v^{n+1+2k} = n! \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1}{2k} u^{n+1-2k} v^{n+1+2k}.$$

In particular,

$$D^n(v^2) = v^{n+1} P_n(v,u), \qquad D^n(uv) = v^{n+1} Q_n(v,u).$$

Proof. For $G = \{x \to 2xy, y \to x^2 + y^2\}$, from Theorem 2.10, we have

$$D^n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} a(n,k) x^{2k+1} y^{n-2k} = \sum_{k=0}^{\lfloor n/2 \rfloor} a(n,k)\, xy^n \left(\frac{x}{y}\right)^{2k}.$$

Set $x = v^2$ and $y = uv$. We get

$$D^n(v^2) = \sum_{k=0}^{\lfloor n/2 \rfloor} a(n,k) v^2 (uv)^n \left(\frac{v^2}{uv}\right)^{2k} = \sum_{k=0}^{\lfloor n/2 \rfloor} a(n,k) u^{n-2k} v^{n+2+2k}.$$

Finally, it is straightforward to see that the new grammar $\{u \to v^3, v \to uv^2\}$ can be derived from the previous grammar $\{x \to 2xy, y \to x^2 + y^2\}$ used in Theorem 2.10, via the substitutions $x = v^2$ and $y = uv$.

By equating the two expressions for $D(x)$ obtained from different perspectives,

$$D(x) = D(v^2) = 2v\, D(v), \qquad D(x) = 2xy = 2uv^3,$$

we solve for $D(v)$ and obtain $D(v) = uv^2$.

Similarly, to determine $D(u)$, we consider the two expressions

$$D(y) = D(uv) = u\, D(v) + v\, D(u) = u^2 v^2 + v\, D(u)$$

and

$$D(y) = x^2 + y^2 = v^4 + u^2 v^2.$$

Solving this equation for $D(u)$ yields $D(u) = v^3$.

Similarly, we can get the expression of $D^n(uv)$. $\square$

Although there are several grammars can generate $a(n,k)$ and $b(n,k)$, we will adopt the grammar $G = \{x \to 2xy, y \to x^2 + y^2\}$ in the deduction of subsequent results. The interested readers can explore other related grammars later.

Another powerful application of the context-free grammar is to derive a lot of convolution formulas.

Theorem 2.12. *For any nonnegative integer n, we have*

$$P_{n+1}(x,y) = 2\sum_{k=0}^{n} \binom{n}{k} P_k(x,y) Q_{n-k}(x,y)$$

and

$$Q_{n+1}(x,y) = \sum_{k=0}^{n} \binom{n}{k} (P_k(x,y) P_{n-k}(x,y) + Q_k(x,y) Q_{n-k}(x,y)).$$

Proof. Let $G = \{x \to 2xy, y \to x^2 + y^2\}$. By Leibniz's formula,

$$D^{n+1}(x) = 2D^n(xy) = 2\sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k}(y),$$

equivalently,

$$P_{n+1}(x,y)=2\sum_{k=0}^{n}\binom{n}{k}P_k(x,y)Q_{n-k}(x,y).$$

Similarly, from

$$\begin{aligned}D^{n+1}(y)&=D^n(x^2+y^2)\\&=D^n(x^2)+D^n(y^2)\\&=\sum_{k=0}^{n}\binom{n}{k}D^k(x)D^{n-k}(x)+\sum_{k=0}^{n}\binom{n}{k}D^k(y)D^{n-k}(y)\\&=\sum_{k=0}^{n}\binom{n}{k}(D^k(x)D^{n-k}(x)+D^k(y)D^{n-k}(y)),\end{aligned}$$

we can get

$$Q_{n+1}(x,y)=\sum_{k=0}^{n}\binom{n}{k}(P_k(x,y)P_{n-k}(x,y)+Q_k(x,y)Q_{n-k}(x,y)).$$

The proof is completed. □

The above theorem can be rephrased as the following form, by extracting the coefficients of $x^{2t+1}y^{n+1-2t}$ ($x^{2t}y^{n+2-2t}$, respectively) on both sides of the equation about $P_{n+1}(x,y)$ ($Q_{n+1}(x,y)$, respectively) in Theorem 2.12.

Theorem 2.13. *For any nonnegative integer n, we have*

$$(n+1)\binom{n+2}{2t+1}=2\sum_{k=0}^{n}\sum_{\substack{i,j\\ i+j=t}}\binom{k+1}{2i+1}\binom{n-k+1}{2j}$$

and

$$t\binom{n+3}{2t+1}=\sum_{k=0}^{n}\left(\sum_{\substack{i,j\\ i+j=t-1}}\binom{k+1}{2i+1}\binom{n-k+1}{2j+1}+\sum_{\substack{i,j\\ i+j=t}}\binom{k+1}{2i}\binom{n-k+1}{2j}\right).$$

Similar to the proof of Theorem 2.12, we are able to derive a mutual recurrence relations between $P_n(x,y)$ and $Q_n(x,y)$.

Theorem 2.14. *For any nonnegative integer n, we have*

$$y^{n+1}P_n(x,y)=xn!\sum_{k=0}^{n}\frac{y^k(y^2-x^2)^{n-k}}{k!}Q_k(x,y),$$

and

$$y^{n+1}Q_n(x,y)=n!\,y^2(y^2-x^2)^n+xn!\sum_{k=1}^{n}\frac{y^k(y^2-x^2)^{n-k}}{k!}P_k(x,y).$$

Proof. Let $G=\{x\to 2xy,\ y\to x^2+y^2\}$. We have known in Theorem 2.10 that $D^n(x)=P_n(x,y)$ and $D^n(y)=Q_n(x,y)$. In addition, it is not hard to get that

$$D^n\left(\frac{x}{y}\right)=\frac{n!x(y^2-x^2)^n}{y^{n+1}}$$

for any nonnegative integer n, and

$$D\left(\frac{y^2}{x}\right)=2xy=D(x),$$

thus

$$D^n\left(\frac{y^2}{x}\right) = P_n(x, y)$$

for any positive integer n.

Now let us utilize Leibniz's formula. On one hand, we can derive that

$$\begin{aligned}
D^n(x) &= D^n\left(y \cdot \frac{x}{y}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} D^k(y) D^{n-k}\left(\frac{x}{y}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} D^k(y) \frac{(n-k)!x(y^2-x^2)^{n-k}}{y^{n-k+1}},
\end{aligned}$$

equivalently,

$$y^{n+1} P_n(x, y) = x n! \sum_{k=0}^{n} \frac{y^k (y^2-x^2)^{n-k}}{k!} Q_k(x, y).$$

On the other hand,

$$\begin{aligned}
D^n(y) &= D^n\left(\frac{y^2}{x} \cdot \frac{x}{y}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} D^k\left(\frac{y^2}{x}\right) D^{n-k}\left(\frac{x}{y}\right) \\
&= \frac{y^2}{x} D^n\left(\frac{x}{y}\right) + \sum_{k=1}^{n} \binom{n}{k} D^k\left(\frac{y^2}{x}\right) D^{n-k}\left(\frac{x}{y}\right) \\
&= \frac{n!(y^2-x^2)^n}{y^{n-1}} + \sum_{k=1}^{n} \binom{n}{k} D^k\left(\frac{y^2}{x}\right) \frac{(n-k)!x(y^2-x^2)^{n-k}}{y^{n-k+1}},
\end{aligned}$$

equivalently,

$$y^{n+1} Q_n(x, y) = n!\, y^2 (y^2-x^2)^n + x n! \sum_{k=1}^{n} \frac{y^k (y^2-x^2)^{n-k}}{k!} P_k(x, y).$$

The desired mutual recurrence relations between $P_n(x, y)$ and $Q_n(x, y)$ follow. □

For any letter z in the alphabet A, we can define the exponential generating function, denoted by $\mathrm{Gen}\,(z, t)$, of the formal derivative $D(z)$ as:

$$\mathrm{Gen}\,(z, t) = \sum_{n \geqslant 0} D^n(z) \frac{t^n}{n!},$$

see [4]. The following relations are straightforward:

$$\begin{aligned}
&\mathrm{Gen}'(z, t) = \mathrm{Gen}\,(D(z), t), \\
&\mathrm{Gen}\,(u+v, t) = \mathrm{Gen}\,(u, t) + \mathrm{Gen}\,(v, t), \\
&\mathrm{Gen}\,(uv, t) = \mathrm{Gen}\,(u, t)\mathrm{Gen}\,(v, t),
\end{aligned}$$

where $\mathrm{Gen}'(z, t)$ represents the derivative of $\mathrm{Gen}\,(z, t)$ with respect to t.

Theorem 2.15. *Let $G = \{x \to 2xy,\ y \to x^2 + y^2\}$. Then*

$$\mathrm{Gen}\,(x, t) = \frac{x}{1 - 2yt - (x^2-y^2)t^2}$$

and

$$\mathrm{Gen}\,(y,t)=\frac{y+(x^2-y^2)t}{1-2yt-(x^2-y^2)t^2}.$$

Proof. First we consider $\mathrm{Gen}\,(x,t)$. It is easy to check that

$$D(x^{-1}y)=x-x^{-1}y^2,\ \ D^2(x^{-1}y)=0.$$

So

$$\mathrm{Gen}\,(x^{-1}y,t)=x^{-1}y+(x-x^{-1}y^2)t. \tag{8}$$

Furthermore,

$$\int \mathrm{Gen}\,(x^{-1}y,t)\,\mathrm{d}t=\int (x^{-1}y+(x-x^{-1}y^2)t)\,\mathrm{d}t=x^{-1}yt+\frac{x-x^{-1}y^2}{2}t^2+C \tag{9}$$

for some constant C.

On the other hand, notice that

$$\mathrm{Gen}\,(y,t)=\mathrm{Gen}\,(x^{-1}y,t)\mathrm{Gen}\,(x,t). \tag{10}$$

Then

$$\mathrm{Gen}\,'(x,t)=\mathrm{Gen}\,(2xy,t)=2\mathrm{Gen}\,(x,t)\mathrm{Gen}\,(y,t)=2\mathrm{Gen}\,^2(x,t)\mathrm{Gen}\,(x^{-1}y,t),$$

which implies that

$$\mathrm{Gen}\,(x^{-1}y,t)=\frac{\mathrm{Gen}\,'(x,t)}{2\mathrm{Gen}\,^2(x,t)}.$$

Integrate both sides of the above equation, together with (9), we get

$$-\frac{1}{2\mathrm{Gen}\,(x,t)}=x^{-1}yt+\frac{x-x^{-1}y^2}{2}t^2+C.$$

Rewrite it as

$$2\mathrm{Gen}\,(x,t)\left(x^{-1}yt+\frac{x-x^{-1}y^2}{2}t^2+C\right)=-1.$$

Compare both sides when $t=0$, i.e., $2xC=-1$, we know $C=-\frac{1}{2x}$. That is to say,

$$\mathrm{Gen}\,(x,t)=\frac{x}{1-2yt-(x^2-y^2)t^2}.$$

As a consequence, $\mathrm{Gen}\,(y,t)$ follows from (10) and (8) immediately:

$$\mathrm{Gen}\,(y,t)=(x^{-1}y+(x-x^{-1}y^2)t)\frac{x}{1-2yt-(x^2-y^2)t^2}=\frac{y+(x^2-y^2)t}{1-2yt-(x^2-y^2)t^2}.$$

The proof is completed. □

Recall that

$$D^n(x)=P_n(x,y),\ \ D^n(y)=Q_n(x,y),$$

then the exponential generating functions of $P_n(x,y)$ and $Q_n(x,y)$ follow from Theorem 2.15 directly.

Theorem 2.16. *We have*

$$\sum_{n\geqslant 0}P_n(x,y)\frac{t^n}{n!}=\frac{x}{1-2yt-(x^2-y^2)t^2}$$

and

$$\sum_{n\geqslant 0}Q_n(x,y)\frac{t^n}{n!}=\frac{y+(x^2-y^2)t}{1-2yt-(x^2-y^2)t^2}.$$

The equivalent forms are as below.

Theorem 2.17. *We have*

$$\sum_{n\geqslant 0}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}x^{2k+1}y^{n-2k}t^n=\frac{x}{1-2yt-(x^2-y^2)t^2}$$

and

$$\sum_{n\geqslant 0}\sum_{k=0}^{\lfloor (n+1)/2\rfloor}\binom{n+1}{2k}x^{2k}y^{n+1-2k}t^n=\frac{y+(x^2-y^2)t}{1-2yt-(x^2-y^2)t^2}.$$

3. (Homogeneous) γ-positivity and bi-γ-positivity of $P_n(x)$ and $P_n(x,y)$

The γ-positivity is an important property for the polynomials occurring in combinatorics, which provides a natural approach to study the symmetric and unimodal polynomials. Let us recall the definition of the γ-positivity.

Definition 3.1. Let $f(x)=\sum_{i=0}^n f_ix^i$ be a univariate symmetric polynomial, i.e., $f_i=f_{n-i}$ for any $0\leqslant i\leqslant n$. If $f(x)$ can be expanded as

$$f(x)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\gamma_kx^k(1+x)^{n-2k}$$

with $\gamma_k\geqslant 0$ for $0\leqslant k\leqslant\lfloor\frac{n}{2}\rfloor$, then we say that $f(x)$ is a γ-positive polynomial.

In past few years, the γ-positivity of some famous combinatorial polynomials has been revealed in a lot of publications, e.g., Eulerian polynomials [14], Eulerian polynomials of type B [6,26], derangement polynomials [1], Narayana polynomials of types A and B [23], as well as [17,18].

It is obvious that $P_n(x)$ and $Q_n(x)$ are both symmetric polynomials. In this section, we will show that $P_n(x)$ and $Q_n(x)$ are γ-positive when n is odd, and sum of two γ-positive polynomials when n is even. Before doing that, we need an auxiliary result.

For the grammar

$$G=\{x\to 2xy,\, y\to x^2+y^2\},$$

by setting $v=x^2+y^2$, it is easy to see that $D(v)=4x^2y+2yv$. Based on it, we get a change of grammar:

$$H=\{x\to 2xy,\, y\to v,\, v\to 4x^2y+2yv\}.$$

Lemma 3.1. *Based on the grammar: $H=\{x\to 2xy, y\to v, v\to 4x^2y+2yv\}$, for any $r\geqslant 0$, there exist some integers $\hat{a}(r,k),\hat{b}(r,k),\hat{c}(r,k)$ such that*

$$D^{2r+1}(x)=\sum_{k=0}^{\lfloor r/2\rfloor}\hat{a}(r,k)(xy)^{2k+1}v^{r-2k} \tag{11}$$

and

$$D^{2r}(x)=\sum_{k=0}^{\lfloor (r-1)/2)\rfloor}\hat{b}(r,k)x^{2k+1}y^{2k+2}v^{r-2k-1}+\sum_{k=0}^{\lfloor r/2\rfloor}\hat{c}(r,k)x^{2k+1}y^{2k}v^{r-2k}. \tag{12}$$

In particular, the integers $\hat{a}(r,k),\hat{b}(r,k),\hat{c}(r,k)$ satisfy the mutual recurrence relations:

$$\begin{aligned}
\hat{a}(r,k)&=2(r+k+1)\hat{b}(r,k)+4(r-2k+1)\hat{b}(r,k-1)+2(r+1)\hat{c}(r,k),\\
\hat{b}(r,k)&=2r\hat{a}(r-1,k),\\
\hat{c}(r,k)&=(2k+1)\hat{a}(r-1,k)+4(r+1-2k)\hat{a}(r-1,k-1).
\end{aligned}$$

Proof. We will prove (11) and (12), as well as the recurrence relations for $\hat{a}(r,k)$, $\hat{b}(r,k)$, $\hat{c}(r,k)$, by the induction on r. It is not hard to obtain

n	$D^n(x)$
1	$2xy$
2	$4xy^2+2xv$
3	$24xyv$
4	$96xy^2v+24xv^2+96x^3y^2$
5	$720xyv^2+960x^3y^3$

from which, we get

Index	Values
$(0,0)$	$\hat{a}(0,0)=2$
$(1,0)$	$\hat{a}(1,0)=24,\ \hat{b}(1,0)=4,\ \hat{c}(1,0)=2$
$(2,0)$	$\hat{a}(2,0)=720,\ \hat{b}(2,0)=96,\ \hat{c}(2,0)=24$
$(2,1)$	$\hat{a}(2,1)=960,\ \hat{c}(2,1)=96$

Based on these numerical evidences, we can see that the expressions of $D^{2r+1}(x)$ and $D^{2r}(x)$ (i.e., (11) and (12)), and the recurrence relations of $\hat{a}(r,k), \hat{b}(r,k), \hat{c}(r,k)$ hold for $r=0,1,2$. Assume that they are true for $r-1\geqslant 2$, we will further confirm their validity for r.

Since the result is true for $r-1$, we can get the corresponding $\hat{a}(r-1,k)$ with $k=0,\ldots,\lfloor\frac{r-1}{2}\rfloor$ and $\hat{a}(r-1,k-1)$ with $k=1,\ldots,\lfloor\frac{r+1}{2}\rfloor$. Now set $\hat{b}(r,k)$ and $\hat{c}(r,k)$ as

$$\hat{b}(r,k):=2r\hat{a}(r-1,k) \tag{13}$$

for $k=0,\ldots,\lfloor\frac{r-1}{2}\rfloor$, and

$$\hat{c}(r,k):=(2k+1)\hat{a}(r-1,k)+4(r+1-2k)\hat{a}(r-1,k-1) \tag{14}$$

for $k=0,\ldots,\lfloor\frac{r}{2}\rfloor$. We would like to prove that such $\hat{b}(r,k)$ and $\hat{c}(r,k)$ satisfy the expression of $D^{2r}(x)$ as shown in (12).

Observe that

$$\begin{aligned}
D^{2r}(x) &= D(D^{2r-1}(x))\\
&= D\left(\sum_{k=0}^{\lfloor(r-1)/2\rfloor}\hat{a}(r-1,k)(xy)^{2k+1}v^{r-1-2k}\right)\\
&= \sum_{k=0}^{\lfloor(r-1)/2\rfloor}\hat{a}(r-1,k)D((xy)^{2k+1}v^{r-1-2k})\\
&= \sum_{k=0}^{\lfloor(r-1)/2\rfloor}\hat{a}(r-1,k)\Big(2(2k+1)x^{2k+1}y^{2k+2}v^{r-1-2k}+(2k+1)x^{2k+1}y^{2k}v^{r-2k}\\
&\quad +4(r-1-2k)x^{2k+3}y^{2k+2}v^{r-2-2k}+2(r-1-2k)x^{2k+1}y^{2k+2}v^{r-1-2k}\Big)\\
&= \sum_{k=0}^{\lfloor(r-1)/2\rfloor}\hat{a}(r-1,k)\Big(2rx^{2k+1}y^{2k+2}v^{r-1-2k}+(2k+1)x^{2k+1}y^{2k}v^{r-2k}\\
&\quad +4(r-1-2k)x^{2k+3}y^{2k+2}v^{r-2-2k}\Big).
\end{aligned} \tag{15}$$

On one hand, we extract the coefficient of $x^{2k+1}y^{2k+2}v^{r-2k-1}$ from the right-hand side of (15), which is $2r\hat{a}(r-1,k)$, equal to $\hat{b}(r,k)$ from (13). On the other hand, the coefficient of $x^{2k+1}y^{2k}v^{r-2k}$ on the right-hand side of (15) is

$$(2k+1)\hat{a}(r-1,k)+4(r+1-2k)\hat{a}(r-1,k-1),$$

which is equal to $\hat{c}(r,k)$ from (14). It leads to

$$D^{2r}(x)=\sum_{k=0}^{\lfloor(r-1)/2\rfloor}\hat{b}(r,k)x^{2k+1}y^{2k+2}v^{r-2k-1}+\sum_{k=0}^{\lfloor r/2\rfloor}\hat{c}(r,k)x^{2k+1}y^{2k}v^{r-2k},$$

completing the proof of (12) for r.

Until now, the formula about $D^{2r}(x)$ is settled, let us go ahead to confirm $D^{2r+1}(x)$ as expressed in (11). Based on the known $\hat{b}(r,k)$ with $k=0,\dots,\lfloor\frac{r-1}{2}\rfloor$, $\hat{b}(r,k-1)$ with $k=1,\dots,\lfloor\frac{r+1}{2}\rfloor$, and $\hat{c}(r,k)$ with $k=0,\dots,\lfloor\frac{r}{2}\rfloor$, set

$$\hat{a}(r,k) := 2(r+k+1)\hat{b}(r,k)+4(r-2k+1)\hat{b}(r,k-1)+2(r+1)\hat{c}(r,k) \tag{16}$$

for $k=0,\dots,\lfloor\frac{r}{2}\rfloor$.

It is not hard to get that

$$\begin{aligned}
D^{2r+1}(x) &= D(D^{2r}(x))\\
&= D\left(\sum_{k=0}^{\lfloor (r-1)/2\rfloor}\hat{b}(r,k)x^{2k+1}y^{2k+2}v^{r-2k-1}+\sum_{k=0}^{\lfloor r/2\rfloor}\hat{c}(r,k)x^{2k+1}y^{2k}v^{r-2k}\right)\\
&= \sum_{k=0}^{\lfloor (r-1)/2\rfloor}\hat{b}(r,k)D(x^{2k+1}y^{2k+2}v^{r-2k-1})+\sum_{k=0}^{\lfloor r/2\rfloor}\hat{c}(r,k)D(x^{2k+1}y^{2k}v^{r-2k})\\
&= \sum_{k=0}^{\lfloor (r-1)/2\rfloor}\hat{b}(r,k)\Big(2(2k+1)x^{2k+1}y^{2k+3}v^{r-2k-1}+(2k+2)x^{2k+1}y^{2k+1}v^{r-2k}\\
&\quad +4(r-2k-1)x^{2k+3}y^{2k+3}v^{r-2k-2}+2(r-1-2k)x^{2k+1}y^{2k+3}v^{r-2k-1}\Big)\\
&\quad +\sum_{k=0}^{\lfloor r/2\rfloor}\hat{c}(r,k)\Big(2(2k+1)x^{2k+1}y^{2k+1}v^{r-2k}+2kx^{2k+1}y^{2k-1}v^{r-2k+1}\\
&\quad +4(r-2k)x^{2k+3}y^{2k+1}v^{r-2k-1}+2(r-2k)x^{2k+1}y^{2k+1}v^{r-2k}\Big)\\
&= \sum_{k=0}^{\lfloor (r-1)/2\rfloor}\hat{b}(r,k)\Big(2rx^{2k+1}y^{2k+3}v^{r-2k-1}+(2k+2)x^{2k+1}y^{2k+1}v^{r-2k}\\
&\quad +4(r-2k-1)x^{2k+3}y^{2k+3}v^{r-2k-2}\Big)+\sum_{k=0}^{\lfloor r/2\rfloor}\hat{c}(r,k)\Big(2(r+1)x^{2k+1}y^{2k+1}v^{r-2k}\\
&\quad +2kx^{2k+1}y^{2k-1}v^{r-2k+1}+4(r-2k)x^{2k+3}y^{2k+1}v^{r-2k-1}\Big).
\end{aligned} \tag{17}$$

From (17), we obtain that:

- the coefficient of $x^{2k+1}y^{2k+1}v^{r-2k}$ is

$$(2k+2)\hat{b}(r,k)+4(r-2k+1)\hat{b}(r,k-1)+2(r+1)\hat{c}(r,k);$$

- the coefficient of $x^{2k+1}y^{2k+3}v^{r-2k-1}$ is $2r\hat{b}(r,k)$;
- the coefficient of $x^{2k+3}y^{2k+1}v^{r-2k-1}$ is

$$2(k+1)\hat{c}(r,k+1)+4(r-2k)\hat{c}(r,k).$$

It is worth indicating that if

$$2r\hat{b}(r,k)=2(k+1)\hat{c}(r,k+1)+4(r-2k)\hat{c}(r,k),$$

i.e., the coefficients of $x^{2k+1}y^{2k+3}v^{r-2k-1}$ and $x^{2k+3}y^{2k+1}v^{r-2k-1}$ are equal, then

$$\begin{aligned}
&2r\hat{b}(r,k)x^{2k+1}y^{2k+3}v^{r-2k-1}+(2(k+1)\hat{c}(r,k+1)+4(r-2k)\hat{c}(r,k))x^{2k+3}y^{2k+1}v^{r-2k-1}\\
&=2r\hat{b}(r,k)(x^{2k+1}y^{2k+3}v^{r-2k-1}+x^{2k+3}y^{2k+1}v^{r-2k-1})\\
&=2r\hat{b}(r,k)x^{2k+1}y^{2k+1}(x^2+y^2)v^{r-2k-1}\\
&=2r\hat{b}(r,k)x^{2k+1}y^{2k+1}v^{r-2k},
\end{aligned}$$

which gives one more coefficient, that is $2r\hat{b}(r,k)$, of $x^{2k+1}y^{2k+1}v^{r-2k}$. In all, the coefficient of $x^{2k+1}y^{2k+1}v^{r-2k}$ is

$$
\begin{aligned}
&(2k+2)\hat{b}(r,k)+4(r-2k+1)\hat{b}(r,k-1)+2(r+1)\hat{c}(r,k)+2r\hat{b}(r,k)\\
&=2(r+k+1)\hat{b}(r,k)+4(r-2k+1)\hat{b}(r,k-1)+2(r+1)\hat{c}(r,k)\\
&=\hat{a}(r,k),
\end{aligned}
$$

where (16) is used in the last equation. So the expression (11) for $D^{2r+1}(x)$ follows.

We are left to show that

$$2r\hat{b}(r,k)=2(k+1)\hat{c}(r,k+1)+4(r-2k)\hat{c}(r,k) \tag{18}$$

is true. Actually, it is equivalent to a (horizontal) recurrence relation of $\hat{a}(r,k)$:

$$
\begin{aligned}
(k+1)(2k+3)\hat{a}(r-1,k+1)=2((r-1-4k)(r-2)+8k^2)\hat{a}(r-1,k)\\
-8(r-2k)(r+1-2k)\hat{a}(r-1,k-1),
\end{aligned} \tag{19}
$$

by substituting the following relations (recall them from (13) and (14)) into (18):

$$
\begin{aligned}
\hat{b}(r,k)&=2r\hat{a}(r-1,k),\\
\hat{c}(r,k)&=(2k+1)\hat{a}(r-1,k)+4(r+1-2k)\hat{a}(r-1,k-1),\\
\hat{c}(r,k+1)&=(2k+3)\hat{a}(r-1,k+1)+4(r-1-2k)\hat{a}(r-1,k).
\end{aligned}
$$

We will prove (19) by the induction on r. When $r=1,2$, it can be verified trivially, from

$$\hat{a}(0,0)=2 \qquad \text{and} \qquad \hat{a}(1,0)=24.$$

Now assume that (19) holds for $r-1\geqslant 1$, thus we have

$$
\begin{aligned}
(k+1)(2k+3)\hat{a}(r-2,k+1)=2((r-2-4k)(r-3)+8k^2)\hat{a}(r-2,k)\\
-8(r-1-2k)(r-2k)\hat{a}(r-2,k-1)
\end{aligned} \tag{20}
$$

and

$$
\begin{aligned}
k(2k+1)\hat{a}(r-2,k)=2((r+2-4k)(r-3)+8(k-1)^2)\hat{a}(r-2,k-1)\\
-8(r+1-2k)(r-2k+2)\hat{a}(r-2,k-2).
\end{aligned} \tag{21}
$$

Notice that the setting of $\hat{a}(r,k)$ (see (16)):

$$\hat{a}(r,k)=2(r+k+1)\hat{b}(r,k)+4(r-2k+1)\hat{b}(r,k-1)+2(r+1)\hat{c}(r,k)$$

is equivalent to a (vertical) recurrence relation of $\hat{a}(r,k)$ by involving with (13) and (14):

$$
\begin{aligned}
\hat{a}(r,k)=2(2r(r+k+1)+(r+1)(2k+1))\hat{a}(r-1,k)\\
+8(r-2k+1)(2r+1)\hat{a}(r-1,k-1).
\end{aligned} \tag{22}
$$

Substitute r into $r-1$, and k into $k+1$ or $k-1$ in (22), we have

$$
\begin{aligned}
\hat{a}(r-1,k+1)&=2\big(2(r-1)(r+k+1)+r(2k+3)\big)\hat{a}(r-2,k+1)\\
&\quad+8(r-2k-2)(2r-1)\hat{a}(r-2,k),\\
\hat{a}(r-1,k)&=2\big(2(r-1)(r+k)+r(2k+1)\big)\hat{a}(r-2,k)\\
&\quad+8(r-2k)(2r-1)\hat{a}(r-2,k-1),\\
\hat{a}(r-1,k-1)&=2\big(2(r-1)(r+k-1)+r(2k-1)\big)\hat{a}(r-2,k-1)\\
&\quad+8(r-2k+2)(2r-1)\hat{a}(r-2,k-2).
\end{aligned}
$$

Furthermore, substitute them into (19), it leads to an equivalent form of (19):

$$
\begin{aligned}
&(k+1)(2k+3)A\hat{a}(r-2,k+1)\\
&=B\hat{a}(r-2,k)+C\hat{a}(r-2,k-1)+D\hat{a}(r-2,k-2),
\end{aligned} \tag{23}
$$

where

$$\begin{aligned}
A &= 2\big(2(r-1)(r+k+1)+r(2k+3)\big),\\
B &= 4\big((r-1-4k)(r-2)+8k^2\big)\big(2(r-1)(r+k)+r(2k+1)\big)\\
&\quad -8(k+1)(2k+3)(r-2k-2)(2r-1),\\
C &= 16\big((r-1-4k)(r-2)+8k^2\big)(r-2k)(2r-1)\\
&\quad -16(r-2k)(r+1-2k)\big(2(r-1)(r+k-1)+r(2k-1)\big),\\
D &= -64(r-2k)(r+1-2k)(r-2k+2)(2r-1).
\end{aligned}$$

Substitute (20) into (23), then (23) would be reduced to

$$\bar{B}\hat{a}(r-2,k)=\bar{C}\hat{a}(r-2,k-1)+D\hat{a}(r-2,k-2), \tag{24}$$

where

$$\begin{aligned}
\bar{B} &= 2((r-2-4k)(r-3)+8k^2)A-B\\
&= 4((r-2-4k)(r-3)+8k^2)(2(r-1)(r+k+1)+r(2k+3))\\
&\quad -4((r-1-4k)(r-2)+8k^2)(2(r-1)(r+k)+r(2k+1))\\
&\quad +8(k+1)(2k+3)(r-2k-2)(2r-1)\\
&= 8(r-2k)(2r-1)k(2k+1)
\end{aligned}$$

and

$$\begin{aligned}
\bar{C} &= 8(r-1-2k)(r-2k)A+C\\
&= 16(r-1-2k)(r-2k)(2(r-1)(r+k+1)+r(2k+3))\\
&\quad +16((r-1-4k)(r-2)+8k^2)(r-2k)(2r-1)\\
&\quad -16(r-2k)(r+1-2k)(2(r-1)(r+k-1)+r(2k-1))\\
&= 16(r-2k)(2r-1)((r-3)(r+2-4k)+8(k-1)^2).
\end{aligned}$$

Observe that $\bar{B}, \bar{C}, D$ share the common factor $8(r-2k)(2r-1)$, after eliminating it, (24) reduces to

$$\begin{aligned}
k(2k+1)\hat{a}(r-2,k) &= 2((r-3)(r+2-4k)+8(k-1)^2)\hat{a}(r-2,k-1)\\
&\quad -8(r+1-2k)(r-2k+2)\hat{a}(r-2,k-2),
\end{aligned}$$

which has been claimed in (21).

At this stage, we complete the proof of (19), and so do the proof of (18). Therefore, the desired results follow finally. □

Remark 3.1. On one hand, all the coefficients of the recurrence relations of $\hat{a}(r,k)$, $\hat{b}(r,k)$, $\hat{c}(r,k)$ are nonnegative. On the other hand, the initial conditions of $\hat{a}(r,k)$, $\hat{b}(r,k)$, $\hat{c}(r,k)$ (with small r and k) are also nonnegative. Thus we can confirm the nonnegativity of $\hat{a}(r,k)$, $\hat{b}(r,k)$, $\hat{c}(r,k)$.

In the following, we list some $D^n(x)$ with small n, $1 \leqslant n \leqslant 9$,

n	$D^n(x)$
1	$2xy$
2	$4xy^2+2xv$
3	$24xyv$
4	$96xy^2v+24xv^2+96x^3y^2$
5	$720xyv^2+960x^3y^3$
6	$4320xy^2v^2+5760x^3y^4+720xv^3+8640x^3y^2v$
7	$40320xyv^3+161280x^3y^3v$
8	$322560xy^2v^3+1290240x^3y^4v+40320xv^4+967680x^3y^2v^2+645120x^5y^4$
9	$3628800xyv^4+29030400x^3y^3v^2+11612160x^5y^5$

from which, we get the corresponding $\hat{a}(r,k), \hat{b}(r,k), \hat{c}(r,k)$ as follows:

Index	Values
$(0,0)$	$\hat{a}(0,0)=2$
$(1,0)$	$\hat{a}(1,0)=24,\ \hat{b}(1,0)=4,\ \hat{c}(1,0)=2$
$(2,0)$	$\hat{a}(2,0)=720,\ \hat{b}(2,0)=96,\ \hat{c}(2,0)=24$
$(2,1)$	$\hat{a}(2,1)=960,\ \hat{c}(2,1)=96$
$(3,0)$	$\hat{a}(3,0)=40320,\ \hat{b}(3,0)=4320,\ \hat{c}(3,0)=720$
$(3,1)$	$\hat{a}(3,1)=161280,\ \hat{b}(3,1)=5760,\ \hat{c}(3,1)=8640$
$(4,0)$	$\hat{a}(4,0)=3628800,\ \hat{b}(4,0)=322560,\ \hat{c}(4,0)=40320$
$(4,1)$	$\hat{a}(4,1)=29030400,\ \hat{b}(4,1)=1290240,\ \hat{c}(4,1)=967680$
$(4,2)$	$\hat{a}(4,2)=11612160,\ \hat{c}(4,2)=645120$

Write them as triangular arrays:

$$\hat{a}(r,k):\quad \begin{array}{ccccccc} & & & 2 & & & \\ & & & 24 & & & \\ & & 720 & & 960 & & \\ & & 40320 & & 161280 & & \\ & 3628800 & & 29030400 & & 11612160 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

$$\hat{b}(r,k):\quad \begin{array}{ccccc} & & 4 & & \\ & & 96 & & \\ & 4320 & & 5760 & \\ & 322560 & & 1290240 & \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

$$\hat{c}(r,k):\quad \begin{array}{ccccccc} & & & 2 & & & \\ & & 24 & & 96 & & \\ & & 720 & & 8640 & & \\ & 40320 & & 967680 & & 645120 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

Now we can confirm the γ-positivity of $P_n(x)$.

Theorem 3.1. *If n is odd, then $P_n(x)$ is γ-positive. If n is even, then $P_n(x)$ can be represented as the sum of two γ-positive polynomials.*

Proof. We partition the proof into two parts, according to the parity of n.
Case 1. n is odd.

In this case, assume that $n=2r+1$. On one hand, by Theorem 2.10, together with (1),

$$D^{2r+1}(x)|_{y=1}=P_{2r+1}(x,1)=x\,P_{2r+1}(x^2).$$

On the other hand, from Lemma 3.1,

$$D^{2r+1}(x)|_{y=1}=x\sum_{k=0}^{\lfloor r/2\rfloor}\hat{a}(r,k)x^{2k}(1+x^2)^{r-2k},$$

by noting that $v=x^2+y^2=x^2+1$. Therefore,

$$P_{2r+1}(x)=\sum_{k=0}^{\lfloor r/2\rfloor}\hat{a}(r,k)x^{k}(1+x)^{r-2k}.$$

Recall that $\hat{a}(r,k)$ is nonnegative, thus the γ-positivity of $P_{2r+1}(x)$ follows.
Case 2. n is even.

This time we set $n=2r$. As in Case 1, we have

$$D^{2r}(x)|_{y=1}=P_{2r}(x,1)=x\,P_{2r}(x^2).$$

Further, from Lemma 3.1,

$$D^{2r}(x)|_{y=1}=x\sum_{k=0}^{\lfloor (r-1)/2\rfloor}\hat{b}(r,k)x^{2k}(1+x^2)^{r-2k-1}+x\sum_{k=0}^{\lfloor r/2\rfloor}\hat{c}(r,k)x^{2k}(1+x^2)^{r-2k}.$$

Therefore,

$$P_{2r}(x)=\sum_{k=0}^{\lfloor (r-1)/2\rfloor}\hat{b}(r,k)x^k(1+x)^{r-2k-1}+\sum_{k=0}^{\lfloor r/2\rfloor}\hat{c}(r,k)x^k(1+x)^{r-2k}.$$

Observe that $\hat{b}(r,k)$ and $\hat{c}(r,k)$ are both nonnegative, it means that

$$\sum_{k=0}^{\lfloor (r-1)/2\rfloor}\hat{b}(r,k)\,x^k(1+x)^{r-2k-1}\qquad\text{and}\qquad\sum_{k=0}^{\lfloor r/2\rfloor}\hat{c}(r,k)\,x^k(1+x)^{r-2k}$$

are both γ-positive. So $P_{2r}(x)$ is the sum of two γ-positive polynomials. $\square$

Based on

$$P_{2r+1}(x)=\sum_{k=0}^{\lfloor r/2\rfloor}\hat{a}(r,k)x^k(1+x)^{r-2k}$$

and

$$P_{2r}(x)=\sum_{k=0}^{\lfloor (r-1)/2\rfloor}\hat{b}(r,k)x^k(1+x)^{r-2k-1}+\sum_{k=0}^{\lfloor r/2\rfloor}\hat{c}(r,k)x^k(1+x)^{r-2k},$$

let us extract the coefficients of x^t, we can attain the following two identities.

Theorem 3.2. *We have*

$$(2r+1)!\binom{2r+2}{2t+1}=\sum_{k=0}^{\lfloor r/2\rfloor}\binom{r-2k}{t-k}\hat{a}(r,k),$$

and

$$(2r)!\binom{2r+1}{2t+1}=\sum_{k=0}^{\lfloor (r-1)/2\rfloor}\binom{r-2k-1}{t-k}\hat{b}(r,k)\;+\;\sum_{k=0}^{\lfloor r/2\rfloor}\binom{r-2k}{t-k}\hat{c}(r,k).$$

Worth mentioning, the second identity does not hold in the trivial case $r=t=0$.

The concept of homogeneous γ-positive polynomials was proposed in [23] as an extension of the γ-positive polynomials.

Definition 3.2. Let $f(x,y)$ be a bivariate polynomial. If $f(x,y)$ can be expanded as

$$f(x,y)=\sum_{k=0}^{\lfloor n/2\rfloor}\gamma_k(xy)^k(x+y)^{n-2k}$$

with $\gamma_k\geqslant 0$ for all $0\leqslant k\leqslant\lfloor n/2\rfloor$, then we say that $f(x,y)$ is a homogeneous γ-positive polynomial.

Clearly, when $y=1$, Definition 3.2 is reduced to Definition 3.1.
From (11), we can see that

$$\frac{P_{2r+1}(\sqrt{x},\sqrt{y})}{\sqrt{x}\sqrt{y}}=\sum_{k=0}^{\lfloor r/2\rfloor}\hat{a}(r,k)(xy)^k(x+y)^{r-2k},$$

that is to say,

$$\frac{P_{2r+1}(\sqrt{x},\sqrt{y})}{\sqrt{x}\sqrt{y}}=\sum_{k=0}^{r}a(2r+1,k)x^ky^{r-k}=y^rP_{2r+1}\left(\frac{x}{y}\right)$$

is a homogeneous γ-positive polynomial.
Recall the definition of bi-γ-positive proposed in [24].

Definition 3.3. Let $f(x)$ be a polynomial of degree n with a unique symmetric decomposition

$$f(x) = a(x) + x \cdot b(x),$$

where $a(x)$ and $b(x)$ are symmetric polynomials such that $a(x) = x^n a(\frac{1}{x})$ and $b(x) = x^{n-1} b(\frac{1}{x})$. If $a(x)$ and $b(x)$ are both γ-positive, then we say that $f(x)$ is bi-γ-positive.

Now we extend the concept of bi-γ-positive to bivariate version, called homogeneous bi-γ-positive.

Definition 3.4. Let $f(x, y)$ be a bivariate polynomial such that

$$f(x, y) = a(x, y) + x \cdot b(x, y).$$

If $a(x, y)$ and $b(x, y)$ are both homogeneous γ-positive polynomials, then we say that $f(x, y)$ is homogeneous bi-γ-positive.

From (12),

$$\frac{P_{2r}(\sqrt{x}, \sqrt{y})}{\sqrt{x}} = y \sum_{k=0}^{\lfloor (r-1)/2 \rfloor} \hat{b}(r, k)(xy)^k (x+y)^{r-2k-1} + \sum_{k=0}^{\lfloor r/2 \rfloor} \hat{c}(r, k)(xy)^k (x+y)^{r-2k},$$

thus

$$\frac{P_{2r}(\sqrt{x}, \sqrt{y})}{\sqrt{x}} = \sum_{k=0}^{r} a(2r, k) x^k y^{r-k} = y^r P_{2r}\left(\frac{x}{y}\right)$$

is a homogeneous bi-γ-positive polynomial.

Theorem 3.3. *For any nonnegative integer n, $y^{\lfloor n/2 \rfloor} P_n\left(\frac{x}{y}\right)$ is a homogeneous γ-positive polynomial when n is odd, and a homogeneous bi-γ-positive polynomial when n is even.*

The corresponding results for $Q_n(x)$ and $Q_n(x, y)$ can be deduced in an analogous way, for the sake of brevity, we omit them here.

4. Determinantal expressions

It is known that some important combinatorial statistics can be expressed as the determinants of some matrices, e.g., derangement numbers and polynomials [8,9], Eulerian polynomials and their q-analogues [7], Bernoulli polynomials [27]. In this section, we will give two determinantal expressions of $P_n(x, y)$ and $Q_n(x, y)$, which are actually some particular tridiagonal determinants. In particular, when a matrix of order 0 occurs, we require that its determinant is equal to 1.

Theorem 4.1. *For any nonnegative integer n, the polynomial $P_n(x, y)$ can be expressed into the following tridiagonal determinant of order n:*

$$P_n(x, y) = x \begin{vmatrix} 2y & -1 & 0 & \cdots & 0 & 0 \\ 2(x^2 - y^2) & 4y & -1 & \cdots & 0 & 0 \\ 0 & 6(x^2 - y^2) & 6y & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2(n-1)y & -1 \\ 0 & 0 & 0 & \cdots & n(n-1)(x^2 - y^2) & 2ny \end{vmatrix}.$$

Proof. Let

$$A_n(x, y) := \begin{vmatrix} 2y & -1 & 0 & \cdots & 0 & 0 \\ 2(x^2 - y^2) & 4y & -1 & \cdots & 0 & 0 \\ 0 & 6(x^2 - y^2) & 6y & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2(n-1)y & -1 \\ 0 & 0 & 0 & \cdots & n(n-1)(x^2 - y^2) & 2ny \end{vmatrix}.$$

By expanding $A_n(x,y)$ along the last row, we get the recurrence relation

$$A_n(x,y)=2ny\cdot A_{n-1}(x,y)+n(n-1)(x^2-y^2)\cdot A_{n-2}(x,y).$$

Recall from Theorem 2.8 that the recurrence relation of $P_n(x,y)$:

$$P_n(x,y)=2ny\cdot P_{n-1}(x,y)+n(n-1)(x^2-y^2)\cdot P_{n-2}(x,y).$$

Thus $P_n(x,y)$ and $A_n(x,y)$, so and $xA_n(x,y)$, have the same recurrence relation.

Moreover, the initial conditions of $P_n(x,y)$ and $xA_n(x,y)$ are the same, from

$$P_0(x,y)=x\,A_0(x,y)=x \qquad \text{and} \qquad P_1(x,y)=x\,A_1(x,y)=2xy.$$

Finally, we are able to conclude with $P_n(x,y)=xA_n(x,y)$. $\square$

Let us give one illustrative example for Theorem 4.1. When $n=5$, we have

$$\begin{aligned}P_5(x,y)=&x\begin{vmatrix} 2y & -1 & 0 & 0 & 0\\ 2(x^2-y^2) & 4y & -1 & 0 & 0\\ 0 & 6(x^2-y^2) & 6y & -1 & 0\\ 0 & 0 & 12(x^2-y^2) & 8y & -1\\ 0 & 0 & 0 & 20(x^2-y^2) & 10y\end{vmatrix}\\ =&720xy^5+2400x^3y^3+720x^5y.\end{aligned}$$

Similar to Theorem 4.1, we can obtain the following determinantal expression for $Q_n(x,y)$.

Theorem 4.2. *For any nonnegative integer n, the polynomial $Q_n(x,y)$ can be expressed into the following tridiagonal determinant of order $n+1$:*

$$Q_n(x,y)=\begin{vmatrix} y & -1 & 0 & 0 & \cdots & 0 & 0\\ x^2-y^2 & 2y & -1 & 0 & \cdots & 0 & 0\\ 0 & 2(x^2-y^2) & 4y & -1 & \cdots & 0 & 0\\ 0 & 0 & 6(x^2-y^2) & 6y & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 2(n-1)y & -1\\ 0 & 0 & 0 & 0 & \cdots & n(n-1)(x^2-y^2) & 2ny\end{vmatrix}.$$

Remark 4.1. It is interesting to see that after deleting the first row and the first column of the determinant corresponding to $Q_n(x,y)$, we would revisit the determinant corresponding to $P_n(x,y)$ given in Theorem 4.1.

For example, we have

$$\begin{aligned}Q_5(x,y)=&\begin{vmatrix} y & -1 & 0 & 0 & 0 & 0\\ x^2-y^2 & 2y & -1 & 0 & 0 & 0\\ 0 & 2(x^2-y^2) & 4y & -1 & 0 & 0\\ 0 & 0 & 6(x^2-y^2) & 6y & -1 & 0\\ 0 & 0 & 0 & 12(x^2-y^2) & 8y & -1\\ 0 & 0 & 0 & 0 & 20(x^2-y^2) & 10y\end{vmatrix}\\ =&120x^6+1800x^4y^2+1800x^2y^4+120y^6.\end{aligned}$$

In addition to the above two determinantal expressions each of which is just for $P_n(x,y)$ and $Q_n(x,y)$ independently, we have also the mutual determinantal expressions between $P_n(x,y)$ and $Q_n(x,y)$. We will use Theorems 2.9 and 2.14 for proving Theorems 4.3 and 4.4, respectively. Basically, their proofs are similar to Theorem 4.1, thus we do not present the detailed proofs again.

Theorem 4.3. *For any nonnegative integer n, the polynomial $P_n(x,y)$ can be expressed into the following determinant of order $n+1$, involving in $Q_0(x,y), Q_1(x,y),\ldots,Q_n(x,y)$:*

$$y \cdot P_n(x,y) = (-1)^n x \begin{vmatrix} Q_0(x,y) & 1 & 0 & \cdots & 0 & 0 \\ Q_1(x,y) & \frac{x^2-y^2}{y} & 1 & \cdots & 0 & 0 \\ Q_2(x,y) & 0 & \frac{2(x^2-y^2)}{y} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ Q_{n-1}(x,y) & 0 & 0 & \cdots & \frac{(n-1)(x^2-y^2)}{y} & 1 \\ Q_n(x,y) & 0 & 0 & \cdots & 0 & \frac{n(x^2-y^2)}{y} \end{vmatrix}.$$

For example,

$$\begin{aligned} &P_5(x,y) \\ =&(-1)^5 \frac{x}{y} \begin{vmatrix} y & 1 & 0 & 0 & 0 & 0 \\ x^2+y^2 & \frac{x^2-y^2}{y} & 1 & 0 & 0 & 0 \\ 6x^2y+2y^3 & 0 & \frac{2(x^2-y^2)}{y} & 1 & 0 & 0 \\ 6x^4+36x^2y^2+6y^4 & 0 & 0 & \frac{3(x^2-y^2)}{y} & 1 & 0 \\ 120x^4y+240x^2y^3+24y^5 & 0 & 0 & 0 & \frac{4(x^2-y^2)}{y} & 1 \\ 120x^6+1800x^4y^2+1800x^2y^4+120y^6 & 0 & 0 & 0 & 0 & \frac{5(x^2-y^2)}{y} \end{vmatrix} \\ =&720x^5y+2400x^3y^3+720xy^5. \end{aligned}$$

Theorem 4.4. *For any positive integer n, the polynomial $P_n(x,y)$ can be expressed into the following determinant of order $n+1$, involving in $Q_0(x,y), Q_1(x,y), \ldots, Q_n(x,y)$:*

$$\begin{aligned} &x \cdot P_n(x,y) \\ =&(-1)^n y \begin{vmatrix} Q_0(x,y) & 1 & 0 & \cdots & 0 & 0 \\ Q_1(x,y) & \frac{y^2-x^2}{y} & 1 & \cdots & 0 & 0 \\ Q_2(x,y) & \frac{2(y^2-x^2)^2}{y^2} & \binom{2}{1}\frac{y^2-x^2}{y} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ Q_{n-1}(x,y) & \frac{(n-1)!(y^2-x^2)^{n-1}}{y^{n-1}} & \binom{n-1}{1}\frac{(n-2)!(y^2-x^2)^{n-2}}{y^{n-2}} & \cdots & \binom{n-1}{n-2}\frac{y^2-x^2}{y} & 1 \\ Q_n(x,y) & \frac{n!(y^2-x^2)^n}{y^n} & \binom{n}{1}\frac{(n-1)!(y^2-x^2)^{n-1}}{y^{n-1}} & \cdots & \binom{n}{n-2}\frac{2(y^2-x^2)^2}{y^2} & \binom{n}{n-1}\frac{y^2-x^2}{y} \end{vmatrix}. \end{aligned}$$

For any nonnegative integer n, the polynomial $Q_n(x,y)$ can be expressed into the following matrix of order $n+1$, involving in $P_0(x,y), P_1(x,y), \ldots, P_n(x,y)$:

$$\begin{aligned} &x \cdot Q_n(x,y) \\ =&(-1)^n y \begin{vmatrix} P_0(x,y) & 1 & 0 & \cdots & 0 & 0 \\ P_1(x,y) & \frac{y^2-x^2}{y} & 1 & \cdots & 0 & 0 \\ P_2(x,y) & \frac{2(y^2-x^2)^2}{y^2} & \binom{2}{1}\frac{y^2-x^2}{y} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ P_{n-1}(x,y) & \frac{(n-1)!(y^2-x^2)^{n-1}}{y^{n-1}} & \binom{n-1}{1}\frac{(n-2)!(y^2-x^2)^{n-2}}{y^{n-2}} & \cdots & \binom{n-1}{n-2}\frac{y^2-x^2}{y} & 1 \\ P_n(x,y) & \frac{n!(y^2-x^2)^n}{y^n} & \binom{n}{1}\frac{(n-1)!(y^2-x^2)^{n-1}}{y^{n-1}} & \cdots & \binom{n}{n-2}\frac{2(y^2-x^2)^2}{y^2} & \binom{n}{n-1}\frac{y^2-x^2}{y} \end{vmatrix}. \end{aligned}$$

For example,

$P_4(x,y)$

$$
=(-1)^4\frac{y}{x}\begin{vmatrix} y & 1 & 0 & 0 & 0\\ x^2+y^2 & \frac{y^2-x^2}{y} & 1 & 0 & 0\\ 6x^2y+2y^3 & \frac{2(y^2-x^2)^2}{y^2} & \binom{2}{1}\frac{y^2-x^2}{y} & 1 & 0\\ 6x^4+36x^2y^2+6y^4 & \frac{3!(y^2-x^2)^3}{y^3} & \binom{3}{1}\frac{2(y^2-x^2)^2}{y^2} & \binom{3}{2}\frac{y^2-x^2}{y} & 1\\ 120x^4y+240x^2y^3+24y^5 & \frac{4!(y^2-x^2)^4}{y^4} & \binom{4}{1}\frac{3!(y^2-x^2)^3}{y^3} & \binom{4}{2}\frac{2(y^2-x^2)^2}{y^2} & \binom{4}{3}\frac{y^2-x^2}{y} \end{vmatrix}
$$
$$
=24x^5+240x^3y^2+120xy^4
$$

and

$$
\begin{aligned}
&Q_4(x,y)\\
=&(-1)^4\frac{y}{x}\begin{vmatrix} x & 1 & 0 & 0 & 0\\ 2xy & \frac{y^2-x^2}{y} & 1 & 0 & 0\\ 2x^3+6xy^2 & \frac{2(y^2-x^2)^2}{y^2} & \binom{2}{1}\frac{y^2-x^2}{y} & 1 & 0\\ 24x^3y+24xy^3 & \frac{3!(y^2-x^2)^3}{y^3} & \binom{3}{1}\frac{2(y^2-x^2)^2}{y^2} & \binom{3}{2}\frac{y^2-x^2}{y} & 1\\ 24x^5+240x^3y^2+120xy^4 & \frac{4!(y^2-x^2)^4}{y^4} & \binom{4}{1}\frac{3!(y^2-x^2)^3}{y^3} & \binom{4}{2}\frac{2(y^2-x^2)^2}{y^2} & \binom{4}{3}\frac{y^2-x^2}{y} \end{vmatrix}\\
=&120x^4y+240x^2y^3+24y^5.
\end{aligned}
$$

Recall that

$$
P_n(x,1)=xP_n(x^2),\quad Q_n(x,1)=Q_n(x^2).
$$

Thus we can also get some determinantal expressions of $P_n(x)$ and $Q_n(x)$, as the corollaries of Theorems 4.1, 4.2, 4.3, 4.4.

Corollary 4.1. *For any nonnegative integer n, we have*

$$
P_n(x)=\begin{vmatrix} 2 & -1 & 0 & \cdots & 0 & 0\\ 2(x-1) & 4 & -1 & \cdots & 0 & 0\\ 0 & 6(x-1) & 6 & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 2(n-1) & -1\\ 0 & 0 & 0 & \cdots & n(n-1)(x-1) & 2n \end{vmatrix}.
$$

Corollary 4.2. *For any nonnegative integer n, we have*

$$
Q_n(x)=\begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0\\ x-1 & 2 & -1 & 0 & \cdots & 0 & 0\\ 0 & 2(x-1) & 4 & -1 & \cdots & 0 & 0\\ 0 & 0 & 6(x-1) & 6 & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 2(n-1) & -1\\ 0 & 0 & 0 & 0 & \cdots & n(n-1)(x-1) & 2n \end{vmatrix}.
$$

Corollary 4.3. *For any nonnegative integer n, we have*

$$
P_n(x)=(-1)^n\begin{vmatrix} Q_0(x) & 1 & 0 & \cdots & 0 & 0\\ Q_1(x) & x-1 & 1 & \cdots & 0 & 0\\ Q_2(x) & 0 & 2(x-1) & \ddots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ Q_{n-1}(x) & 0 & 0 & \cdots & (n-1)(x-1) & 1\\ Q_n(x) & 0 & 0 & \cdots & 0 & n(x-1) \end{vmatrix}.
$$

Corollary 4.4. *For any positive integer n,*

$$x\cdot P_n(x)$$
$$=(-1)^n\begin{vmatrix} Q_0(x) & 1 & 0 & \cdots & 0 & 0\\ Q_1(x) & 1-x & 1 & \cdots & 0 & 0\\ Q_2(x) & 2(1-x)^2 & \binom{2}{1}(1-x) & \ddots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ Q_{n-1}(x) & (n-1)!(1-x)^{n-1} & \binom{n-1}{1}(n-2)!(1-x)^{n-2} & \cdots & \binom{n-1}{n-2}(1-x) & 1\\ Q_n(x) & n!(1-x)^n & \binom{n}{1}(n-1)!(1-x)^{n-1} & \cdots & \binom{n}{n-2}2(1-x)^2 & \binom{n}{n-1}(1-x)\end{vmatrix}.$$

For any nonnegative integer n,

$$Q_n(x)$$
$$=(-1)^n\begin{vmatrix} P_0(x) & 1 & 0 & \cdots & 0 & 0\\ P_1(x) & 1-x & 1 & \cdots & 0 & 0\\ P_2(x) & 2(1-x)^2 & \binom{2}{1}(1-x) & \ddots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ P_{n-1}(x) & (n-1)!(1-x)^{n-1} & \binom{n-1}{1}(n-2)!(1-x)^{n-2} & \cdots & \binom{n-1}{n-2}(1-x) & 1\\ P_n(x) & n!(1-x)^n & \binom{n}{1}(n-1)!(1-x)^{n-1} & \cdots & \binom{n}{n-2}2(1-x)^2 & \binom{n}{n-1}(1-x)\end{vmatrix}.$$

5. More combinatorial properties

Actually, the two numbers $a(n,k)$ and $b(n,k)$ also enjoy other common combinatorial properties, such as Pólya-frequency, unimodal, log-concave, real-rooted, asymptotic normality.

First, let us recall the definitions of Pólya-frequency, unimodal, log-concave sequences. Let $a_0, a_1, a_2, \ldots$ be a sequence of nonnegative real numbers.

- The sequence $a_0, a_1, a_2, \ldots$ is called unimodal if
$$a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{k-1} \leqslant a_k \geqslant a_{k+1} \geqslant \cdots$$
for some k.
- The sequence $a_0, a_1, a_2, \ldots$ is called log-concave if $a_{i-1}a_{i+1} \leqslant a_i^2$ for all $i \geqslant 1$.
- The sequence $a_0, a_1, a_2, \ldots$ is called a Pólya-frequency sequence (or a PF sequence for short) if the matrix $(a_{i-j})_{i,j\geqslant 0}$ is a totally positive matrix (all minors of the matrix $(a_{i-j})_{i,j\geqslant 0}$ are nonnegative).

The readers can refer to [19,28] for more information on these subjects. In particular, Wang and Yeh [28] gave a very nice criterion to confirm if a triangular array satisfying some recurrence relation would be a PF sequence.

Lemma 5.1. *[28, Corollary 3] Let $\{T(n,k)\}$ be a triangular array defined by*

$$T(n,k) = (an+bk+c)T(n-1,k-1) + (en+fk+g)T(n-1,k).$$

Suppose that $af \geqslant eb$ and $(a+b+c)f \geqslant (e+g)b$. Then the triangular array $\{T(n,k)\}$ is PF, and thus unimodal and log-concave.

Applying Lemma 5.1 to the triangular arrays $\{a(n,k)\}$ and $\{b(n,k)\}$, by virtue of (3) and (4), we can get the following result immediately.

Theorem 5.1. *The triangular arrays $\{a(n,k)\}$ and $\{b(n,k)\}$ are both PF sequences, and thus unimodal and log-concave.*

Remark 5.1. Notice that all the roots of $P_n(x)$ and $Q_n(x)$ are real, simple, and negative, since $\{a(n,k)\}$ and $\{b(n,k)\}$ are both PF, see also [28].

Assume that X_n is a sequence of random variables with mean μ_n and variance σ_n^2, where $n \in \mathbb{N}$. Let

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t.$$

In the research on combinatorial enumeration, it is frequently found that for each $x \in \mathbb{R}$,

$$\mathrm{Prob}(X_n \leqslant \mu_n + x\sigma_n) \to \mathcal{N}(x),$$

as $n \to \infty$, e.g., Stirling numbers of the second kind [15], Eulerian numbers [2], numbers of cycles (inversions, left-to-right minima, respectively) in a random permutation [13]. Under this situation, we would call X_n is asymptotically normal distributed with mean μ_n and variance σ_n^2.

Let us recall the asymptotic distribution theory for Eulerian-type recurrences established by Hwang et al. [16]. Consider the sequence of functions $A_n(x)$ defined recursively by

$$A_n(x) = \big(\alpha(x)n + \gamma(x)\big)A_{n-1}(x) + \beta(x)(1-x)\, A'_{n-1}(x), \qquad n \geqslant 1, \tag{25}$$

where $A_0(x)$, $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ are given. In particular, when

$$A_0(x) = 1, \qquad \alpha(x) = \beta(x) = x, \qquad \gamma(x) = 1 - x,$$

the function $A_n(x)$ specializes to the classical Eulerian polynomial.

For each $n \geqslant 0$, define the random variable X_n by

$$\mathrm{Prob}(X_n = k) = \frac{[x^k]A_n(x)}{A_n(1)}, \qquad k \geqslant 0, \tag{26}$$

where $[x^k]A_n(x)$ denotes the coefficient of x^k in the polynomial $A_n(x)$.

Lemma 5.2 *(Asymptotic normality of X_n). [16, Theorem 1] Assume that the sequence of functions $A_n(x)$ is defined recursively by (25) satisfying (i) $[x^k]A_n(x) \geqslant 0$ and $A_n(x) \not\equiv 0$ for $k, n \geqslant 0$, and (ii) $A_0(x)$, $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ analytic in $|x| \leqslant 1$. If, furthermore, $\alpha(1) + 2\beta(1) > 0$ and $\sigma^2 > 0$, where*

$$\mu := \frac{\alpha'(1)}{\alpha(1)+\beta(1)} \qquad \text{and} \qquad \sigma^2 := \mu + \frac{\alpha''(1) - 2\mu\beta'(1) - \alpha\mu^2}{\alpha(1)+2\beta(1)},$$

then the sequence of random variables X_n, defined by (26), satisfies $X_n \sim \mathcal{N}(\mu n, \sigma^2 n)$, namely, X_n is asymptotically normally distributed with the mean and the variance asymptotic to μn and $\sigma^2 n$, respectively.

Recall the recurrence relations for $P_n(x)$ and $Q_n(x)$ given in Theorem 2.7:

$$P_n(x) = \big((x+1)n + 1 - x\big)P_{n-1}(x) + 2x(1-x)P'_{n-1}(x)$$

and

$$Q_n(x) = n(x+1)Q_{n-1}(x) + 2x(1-x)Q'_{n-1}(x).$$

As an application of the asymptotic distribution theory for Eulerian-type recurrences from Hwang et al. [16, Theorem 1], by setting

$$\alpha(x) = x + 1, \quad \beta(x) = 2x, \quad \gamma(x) = 1 - x \quad \text{for } P_n(x),$$

and

$$\alpha(x) = x + 1, \quad \beta(x) = 2x, \quad \gamma(x) = 0 \quad \text{for } Q_n(x),$$

we can confirm the asymptotic normality of the sequences $\{a(n,k)\}$ and $\{b(n,k)\}$.

Theorem 5.2. *Each of the sequences $\{a(n,k)\}$ and $\{b(n,k)\}$ is asymptotically normally distributed with the mean and the variance asymptotic to $\frac{n}{4}$ and $\frac{n}{16}$, respectively.*

6. Concluding remarks

In this paper, we propose a new kind of signed permutations and related multivariate polynomials F_n, G_n, H_n, L_n, as well as the univariate polynomials $P_n(x)$ and $Q_n(x)$ and the bivariate polynomials $P_n(x, y)$ and $Q_n(x, y)$ as the generating functions of $a(n,k) = n!\binom{n+1}{2k+1}$ and $b(n,k) = n!\binom{n+1}{2k}$. A lot of combinatorial properties of them are revealed, e.g., recurrence relations, (homogeneous) γ-positivity, (homogeneous) bi-γ-positivity, determinantal expressions, Pólya-frequency, unimodal, log-concave, real-rooted, asymptotic normality.

We believe that there remain many interesting related questions to be explored. For example,

- In Lemma 3.1 we established the mutual recurrence relations and initial conditions for the γ-coefficients $\hat{a}(r,k)$, $\hat{b}(r,k)$, and $\hat{c}(r,k)$ associated with $P_n(x)$ (and similarly for $P_n(x, y)$). A natural next step is to investigate whether these γ-coefficients admit combinatorial models or combinatorial interpretations.
- In this paper we primarily analyze the generating function $L(x, y)$ defined in terms of the number of sign changes. It is therefore natural to further study the combinatorial properties of the generating functions defined in terms of the number of ascents and/or descents.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

References

[1] C.A. Athanasiadis, Gamma-positivity in combinatorics and geometry, Sémin. Lothar. Comb. 77 (2018) B77i.
[2] E.A. Bender, Central and local limit theorems applied to asymptotic enumeration, J. Comb. Theory, Ser. A 15 (1973) 91–111.
[3] W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theor. Comput. Sci. 117 (1993) 113–129.
[4] W.Y.C. Chen, A.M. Fu, Context-free grammars for permutations and increasing trees, Adv. Appl. Math. 82 (2017) 58–82.
[5] W.Y.C. Chen, R.X.J. Hao, H.R.L. Yang, Context-free grammars and multivariate stable polynomials over Stirling permutations, arXiv:1208.1420v2.
[6] C.-O. Chow, On certain combinatorial expansions of the Eulerian polynomials, Adv. Appl. Math. 41 (2008) 133–157.
[7] C.-O. Chow, Some determinantal representations of Eulerian polynomials and their q-analogues, J. Integer Seq. 26 (2023) 23.7.1.
[8] C.-O. Chow, Some determinantal representations of derangement numbers and polynomials, Ramanujan J. 64 (2024) 1243–1259.
[9] C.-O. Chow, Some determinantal representations of derangement polynomials of types B and D, Discrete Math. 347 (2024) 114155.
[10] E.V.F. Conrad, P. Flajolet, The Fermat cubic, elliptic functions, continued fractions and a combinatorial excursion, Sémin. Lothar. Comb. 54 (2005) 1–44.
[11] D. Dumont, Grammaires de William Chen et dérivations dans les arbres et arborescences, Sémin. Lothar. Comb. 37 (1996) 1–21.
[12] D. Dumont, A. Ramamonjisoa, Grammaire de Ramanujan et Arbres de Cayley, Electron. J. Comb. 3 (1996) #R17.
[13] W. Feller, The fundamental limit theorems in probability, Bull. Am. Math. Soc. 51 (1945) 800–832.
[14] D. Foata, M.P. Schützenberger, Théorie géometrique des polynômes eulériens, Lecture Notes in Math., vol. 138, Springer, Berlin, 1970.
[15] L.H. Harper, Stirling behavior is asymptotically normal, Ann. Math. Stat. 38 (1967) 410–414.
[16] H.-K. Hwang, H.-H. Chern, G.-H. Duh, An asymptotic distribution theory for Eulerian recurrences with applications, Adv. Appl. Math. 112 (2020) 101960.
[17] Z. Lin, On γ-positive polynomials arising in pattern avoidance, Adv. Appl. Math. 82 (2017) 1–22.
[18] Z. Lin, J. Zeng, The γ-positivity of basic Eulerian polynomials via group actions, J. Comb. Theory, Ser. A 135 (2015) 112–129.
[19] L.L. Liu, Y. Wang, A unified approach to polynomial sequences with only real zeros, Adv. Appl. Math. 38 (2007) 542–560.
[20] S.-M. Ma, Derivative polynomials and enumeration of permutations by number of interior and left peaks, Discrete Math. 312 (2012) 405–412.
[21] S.-M. Ma, A family of two-variable derivative polynomials for tangent and secant, Electron. J. Comb. 20 (2013), #P11.
[22] S.-M. Ma, Enumeration of permutations by number of alternating runs, Discrete Math. 313 (2013) 1816–1822.
[23] S.-M. Ma, J. Ma, Y.-N. Yeh, γ-positivity and partial γ-positivity of descent-type polynomials, J. Comb. Theory, Ser. A 167 (2019) 257–293.
[24] S.-M. Ma, J. Ma, Y.-N. Yeh, R.R. Zhou, Jacobian elliptic functions and a family of bivariate peak polynomials, Eur. J. Comb. 97 (2021) 103371.
[25] S.-M. Ma, Y.-N. Yeh, Eulerian polynomials, Stirling permutations of the second kind and perfect matchings, Electron. J. Comb. 24 (2017), #P4.27.
[26] T.K. Petersen, Enriched P-partitions and peak algebras, Adv. Math. 209 (2007) 561–610.
[27] F. Qi, R.J. Chapman, Two closed forms for the Bernoulli polynomials, J. Number Theory 159 (2016) 89–100.
[28] Y. Wang, Y.-N. Yeh, Polynomials with real zeros and Pólya frequency sequences, J. Comb. Theory, Ser. A 109 (2005) 63–74.